## Chapter Project

### Following the Trend

Students use what they have learned to evaluate functions.

- Have students choose a function that interests them. For example, students interested in music might find statistics on time between their favorite band's CD releases and the number of CDs sold.
- Once students have decided which function to explore, have them collect data to write a function rule.
- Have students graph the function and specify the domain, range, extrema, $y$-intercept, and zeros. In each case, have students write about the practical implications of these values.
- Have students find the inverse of their function if it exists and write whether the inverse is useful. Students should identify the domain of the inverse and discuss whether it makes sense in the context of the problem.

**KeyVocabulary** Introduce the key vocabulary in the chapter using the routine below.

Define: Two relations are inverse relations if and only if one relation contains the element $(b, a)$ whenever the other relation contains the element $(a, b)$.

Example: The inverse relation of $f(x) = x^2 + 1$ is $y = \pm\sqrt{x - 1}$.

Ask: What is true about the graphs of a function and its inverse relation? The graphs of the function and the inverse relation are symmetric in the line $y = x$.

# CHAPTER 11 Functions from a Calculus Perspective



| Then | Now | Why? |
|---|---|---|
| You analyzed functions from a graphical perspective. | You will:<br>▪ Explore symmetries of graphs.<br>▪ Determine continuity and average rates of change of functions.<br>▪ Use limits to describe end behavior.<br>▪ Find inverse functions algebraically and graphically. | **BUSINESS** Functions are often used throughout the business world. Some of the uses of functions are to analyze costs, predict sales, calculate profit, forecast future costs and revenue, estimate depreciation, and determine the proper labor force.<br>**PREREAD** Create a list of two or three things that you already know about functions.<br>See students' work. |

### Preread

Encourage students to begin their study of the chapter by prereading each lesson. They should think about their background knowledge and make predictions about the content. Allow time for groups to discuss the reading and generate questions. Emphasize the text features such as section headings and Key Concept and Concept Summary Boxes.

# LESSON 11-1 Functions

## 1 Focus

### VerticalAlignment

**Before Lesson 11-1** Solve systems of equations using the properties of real numbers.

**Lesson 11-1** Describe subsets of real numbers.
Identify and evaluate functions and state their domains.

**After Lesson 11-1** Identify the range, $y$-intercept, and zeros of functions.

## 2 Teach

### Scaffolding Questions

Have students read the **Why?** section of the lesson.

**Ask:**

- When can an increase in one value cause another value to increase? Sample answer: Increasing the length of a room will increase the area.
- When can an increase in one value cause another value to decrease? Sample answer: Increasing costs will decrease profits.

**Then**

- You used set notation to denote elements, subsets, and complements.

**Now**

1. Describe subsets of real numbers.
2. Identify and evaluate functions and state their domains.

**Why?**

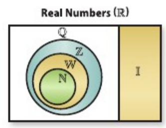
- Many events that occur in everyday life involve two related quantities. For example, to operate a vending machine, you insert money and make a selection. The machine gives you the selected item and any change due. Once your selection is made, the amount of change you receive *depends* on the amount of money you put into the machine.

**NewVocabulary**
set-builder notation
interval notation
function
function notation
independent variable
dependent variable
implied domain
piecewise-defined function
relevant domain

### 1 Describe Subsets of Real Numbers

Real numbers are used to describe quantities such as money and distance. The set of real numbers $\mathbb{R}$ includes the following subsets of numbers.

**KeyConcept** Real Numbers




| Letter | Set | Examples |
|---|---|---|
| $\mathbb{Q}$ | rationals | $0.125, -\frac{7}{8}, \frac{2}{3} = 0.666...$ |
| $\mathbb{I}$ | irrationals | $\sqrt{3} = 1.73205...$ |
| $\mathbb{Z}$ | integers | $-5, 17, -23, 8$ |
| $\mathbb{W}$ | wholes | $0, 1, 2, 3...$ |
| $\mathbb{N}$ | naturals | $1, 2, 3, 4...$ |

These and other sets of real numbers can be described using set-builder notation. **Set-builder notation** uses the properties of the numbers in the set to define the set.

$$\{x \mid -3 \le x \le 16, x \in \mathbb{Z}\}$$

The set of numbers $x$ such that... | $x$ has the given properties... | and $x$ is an element of the given set of numbers.

**Example 1** Use Set Builder Notation

**Describe the set of numbers using set-builder notation.**

**a.** $\{8, 9, 10, 11, ...\}$

The set includes all whole numbers greater than or equal to 8.

$\{x \mid x \ge 8, x \in \mathbb{W}\}$ Read *as the set of all x such that x is greater than or equal to 8 and x is an element of the set of whole numbers.*

**b.** $x < 7$

Unless otherwise stated, you should assume that a given set consists of real numbers. Therefore, the set includes all real numbers less than 7. $\{x \mid x < 7, x \in \mathbb{R}\}$

**c. all multiples of three**

The set includes all integers that are multiples of three. $\{x \mid x = 3n, n \in \mathbb{Z}\}$

**GuidedPractice**

**1A.** $\{1, 2, 3, 4, 5, ....\}$ $\{x \mid x \ge 1, x \in \mathbb{N}\}$

**1B.** $x \le -3$ $\{x \mid x \le -3, x \in \mathbb{R}\}$

**1C.** all multiples of $\pi$ $\{x \mid n\pi, n \in \mathbb{Z}\}$

**StudyTip**

**Look Back** You can review set notation, including unions and intersections of sets.

**Interval notation** uses inequalities to describe subsets of real numbers. The symbols [ or ] are used to indicate that an endpoint is included in the interval, while the symbols ( or ) are used to indicate that an endpoint is not included in the interval. The symbols $\infty$, positive infinity, and $-\infty$, negative infinity, are used to describe the unboundedness of an interval. An interval is *unbounded* if it goes on indefinitely.

| Bounded Intervals | | Unbounded Intervals | |
|---|---|---|---|
| Inequality | Interval Notation | Inequality | Interval Notation |
| $a \le x \le b$ | $[a, b]$ | $x \ge a$ | $[a, \infty)$ |
| $a < x < b$ | $(a, b)$ | $x \le a$ | $(-\infty, a]$ |
| $a \le x < b$ | $[a, b)$ | $x > a$ | $(a, \infty)$ |
| $a < x \le b$ | $(a, b]$ | $x < a$ | $(-\infty, a)$ |
| | | $-\infty < x < \infty$ | $(-\infty, \infty)$ |

**Example 2** Use Interval Notation

**Write each set of numbers using interval notation.**

**a.** $-8 < x \le 16$ $(-8, 16]$

**b.** $x < 11$ $(-\infty, 11)$

**c.** $x \le -16$ or $x > 5$ $(-\infty, -16] \cup (5, \infty)$ $\cup$ read as *union*

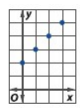
**GuidedPractice**

**2A.** $-4 \le y < -1$ $[-4, -1)$ **2B.** $a \ge -3$ $[-3, \infty)$ **2C.** $x > 9$ or $x < -2$ $(-\infty, -2) \cup (9, \infty)$

## 2 Identify Functions

Recall that a *relation* is a rule that relates two quantities. Such a rule pairs the elements in a set $A$ with elements in a set $B$. The set $A$ of all inputs is the *domain* of the relation, and set $B$ contains all outputs or the *range*.

Relations are commonly represented in four ways.

1. **Verbally** A sentence describes how the inputs and outputs are related.
   *The output value is 2 more than the input value.*
2. **Numerically** A table of values or a set of ordered pairs relates each input ($x$-value) with an output value ($y$-value).
   $\{(0, 2), (1, 3), (2, 4), (3, 5)\}$
3. **Graphically** Points on a graph in the coordinate plane represent the ordered pairs.


4. **Algebraically** An equation relates the $x$- and $y$-coordinates of each ordered pair.
   $y = x + 2$

A **function** is a special type of relation.

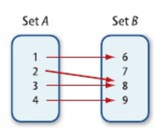
**KeyConcept** Function

Words: A function $f$ from set $A$ to set $B$ is a relation that assigns to each element $x$ in set $A$ *exactly one* element $y$ in set $B$.

Symbols: The relation from set $A$ to set $B$ is a function.

Set $A$ is the domain. $D = \{1, 2, 3, 4\}$

Set $B$ contains the range. $R = \{6, 8, 9\}$




**StudyTip**

**Domain and Range** In this text, the notation for domain and range will be D = and R =, respectively.





- Can an increase in one value lead to both an increase and a decrease in another value? Yes; Sample answer: Increasing production to meet market demand increases profits. Increasing production after the demand is met will reduce profits.

## 1 Describe Subsets of Real Numbers

**Example 1** shows how to describe sets of numbers using set-builder notation. **Example 2** shows how to describe sets of numbers using interval notation.

### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

### Additional Examples

**1** Describe the set of numbers using set-builder notation.

**a.** $\{2, 3, 4, 5, 6, 7\}$
$\{x \mid 2 \le x \le 7, x \in \mathbb{N}\}$

**b.** $x > -17$
$\{x \mid x > -17, x \in \mathbb{R}\}$

**c.** all multiples of seven
$\{x \mid x = 7n, n \in \mathbb{Z}\}$

**2** Write each set of numbers using interval notation.

**a.** $-2 \le x \le 12$ $[-2, 12]$

**b.** $x > -4$ $(-4, \infty)$

**c.** $x < 3$ or $x \ge 54$
$(-\infty, 3) \cup [54, \infty)$

## 2 Identify Functions

**Example 3** shows how to determine whether a relation is a function. **Example 4** shows how to evaluate a function for a given value. **Example 5** shows how to find the domain of a function algebraically. **Example 6** shows how to evaluate a piecewise-defined function for a given value.

### Focus on Mathematical Content

**Interval Notation** The symbols ( or ) are used with a strict inequality, while [ or ] are used when the endpoints are included in the interval. Note that $(a, a)$, $[a, a)$, and $(a, a]$ all are the empty set, while $[a, a]$ is the set $\{a\}$.

## Additional Example

**3** Determine whether each relation represents $y$ as a function of $x$.

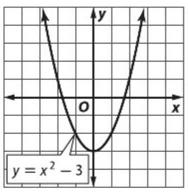
**a.** The input value $x$ is the height of a student in centimeters, and the output value $y$ is the number of books that the student owns. No; there can be more than one $y$-value for an $x$-value.

**b.**

| $x$ | $y$ |
|---|---|
| 1 | −1 |
| 1 | 1 |
| 4 | −2 |
| 4 | 2 |
| 9 | −3 |

No; there is more than one $y$-value for an $x$-value.

**c.**




Yes; there is exactly one $y$-value for each $x$-value.

**d.** $x = 3y^2$ No; there is more than one $y$-value for an $x$-value.

### Teach with Tech

**Interactive Whiteboard** Have students work in groups of two or three to identify two relations that are functions and two relations that are not functions. Have students take turns graphing all four relations on the Interactive WhiteBoard to demonstrate which are functions and which are not. Note that relations that are not functions are best displayed with a scatter plot.

**StudyTip**

**Tabular Method** When a relation fails the vertical line test, an $x$-value has more than one corresponding $y$-value, as shown below.

| $x$ | $y$ |
|---|---|
| −2 | −4 |
| 3 | −1 |
| 3 | 4 |
| 5 | 6 |
| 7 | 9 |

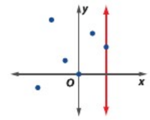
An alternate definition of a function is a set of ordered pairs in which no two different pairs have the same $x$-value. Interpreted graphically, this means that no two points on the graph of a function in the coordinate plane can lie on the same vertical line.

### KeyConcept Vertical Line Test

**Words**

A set of points in the coordinate plane is the graph of a function if each possible vertical line intersects the graph in at most one point.

**Model**




### Example 3 Identify Relations that are Functions

**Determine whether each relation represents $y$ as a function of $x$.**

**a. The input value $x$ is a student's ID number, and the output value $y$ is that student's score on a physics exam.**

Each value of $x$ cannot be assigned to more than one $y$-value. A student cannot receive two different scores on an exam. Therefore, the sentence describes $y$ as a function of $x$.

**StudyTip**

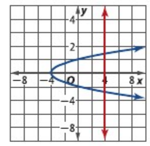
**Functions with Repeated $y$-Values** While a function *cannot* have more than one $y$-value paired with each $x$-value, a function *can* have one $y$-value paired with more than one $x$-value, as shown in Example 3b.

**b.**

| $x$ | $y$ |
|---|---|
| −8 | −5 |
| −5 | −4 |
| 0 | −3 |
| 3 | −2 |
| 6 | −3 |

Each $x$-value is assigned to exactly one $y$-value. Therefore, the table represents $y$ as a function of $x$.

**c.**




A vertical line at $x = 4$ intersects the graph at more than one point. Therefore, the graph does not represent $y$ as a function of $x$.

**d.** $y^2 - 2x = 5$

To determine whether this equation represents $y$ as a function of $x$, solve the equation for $y$.

$$y^2 - 2x = 5 \quad \text{Original equation}$$

$$y^2 = 5 + 2x \quad \text{Add } 2x \text{ to each side.}$$

$$y = \pm\sqrt{5 + 2x} \quad \text{Take the square root of each side.}$$

This equation does not represent $y$ as a function of $x$ because there will be two corresponding $y$-values, one positive and one negative, for any $x$-value greater than −2.5.

### GuidedPractice

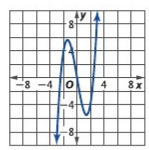
**3A.** The input value $x$ is the area code, and the output value $y$ is a phone number in that area code. **not a function**

**3B.**

| $x$ | $y$ |
|---|---|
| −6 | −7 |
| 2 | 3 |
| 5 | 8 |
| 5 | 9 |
| 9 | 22 |

**not a function**

**3C.**




**function**

**3D.** $3y + 6x = 18$ **function**





### DifferentiatedInstruction AL OL

**Naturalist Learners** Have students find three objects with at least one square face. Have students use sticky notes to label each square with its side length and its area. Record the side measures and areas on the board. Then challenge students to find a function that describes the relation. $A(s) = s^2$

In **function notation**, the symbol $f(x)$ is read *f of x* and interpreted as *the value of the function f at x*. Because $f(x)$ corresponds to the $y$-value of $f$ for a given $x$-value, you can write $y = f(x)$.

| Equation | Related Function |
| --- | --- |
| $y = -6x$ | $f(x) = -6x$ |

Because it can represent any value in the function's domain, $x$ is called the **independent variable**. A value in the range of $f$ is represented by the **dependent variable**, $y$.

**Math HistoryLink**
Leonhard Euler
**(1707–1783)**
A Swiss mathematician, Euler was a prolific mathematical writer, publishing over 800 papers in his lifetime. He also introduced much of our modern mathematical notation, including the use of $f(x)$ for the function $f$.

### Example 4 Find Function Values

**If $g(x) = x^2 + 8x - 24$, find each function value.**

**a.** $g(6)$

To find $g(6)$, replace $x$ with 6 in $g(x) = x^2 + 8x - 24$.

$$\begin{aligned} g(x) &= x^2 + 8x - 24 && \text{Original function} \\ g(6) &= (6)^2 + 8(6) - 24 && \text{Substitute 6 for } x. \\ &= 36 + 48 - 24 && \text{Simplify.} \\ &= 60 && \text{Simplify.} \end{aligned}$$

**b.** $g(-4x)$

$$\begin{aligned} g(x) &= x^2 + 8x - 24 && \text{Original function} \\ g(-4x) &= (-4x)^2 + 8(-4x) - 24 && \text{Substitute } -4x \text{ for } x. \\ &= 16x^2 - 32x - 24 && \text{Simplify.} \end{aligned}$$

**c.** $g(5c + 4)$

$$\begin{aligned} g(x) &= x^2 + 8x - 24 && \text{Original function} \\ g(5c + 4) &= (5c + 4)^2 + 8(5c + 4) - 24 && \text{Substitute } 5c + 4 \text{ for } x. \\ &= 25c^2 + 40c + 16 + 40c + 32 - 24 && \text{Expand } (5c + 4)^2 \text{ and } 8(5c + 4). \\ &= 25c^2 + 80c + 24 && \text{Simplify.} \end{aligned}$$

**GuidedPractice**

If $f(x) = \dfrac{2x + 3}{x^2 - 2x + 1}$, find each function value.

**4A.** $f(12)$ $\dfrac{27}{121}$ **4B.** $f(6x)$ $\dfrac{12x + 3}{36x^2 - 12x + 1}$ **4C.** $f(-3a + 8)$ $\dfrac{-6a + 19}{9a^2 - 42a + 49}$

When you are given a function with an unspecified domain, the **implied domain** is the set of all real numbers for which the expression used to define the function is real. In general, you must exclude values from the domain of a function that would result in division by zero or taking the even root of a negative number.

### Example 5 Find Domains Algebraically

**State the domain of each function.**

**a.** $f(x) = \dfrac{2 + x}{x^2 - 7x}$

When the denominator of $\dfrac{2 + x}{x^2 - 7x}$ is zero, the expression is undefined. Solving $x^2 - 7x = 0$, the excluded values for the domain of this function are $x = 0$ and $x = 7$. The domain of this function is all real numbers except $x = 0$ and $x = 7$, or $\{x \mid x \neq 0, x \neq 7, x \in \mathbb{R}\}$.

**b.** $g(t) = \sqrt{t - 5}$

Because the square root of a negative number cannot be real, $t - 5 \geq 0$. Therefore, the domain of $g(t)$ is all real numbers $t$ such that $t \geq 5$ or $[5, \infty)$.

**StudyTip**

**Naming Functions** You can use other letters to name a function and its independent variable. For example, $f(x) = \sqrt{x - 5}$ and $g(t) = \sqrt{t - 5}$ name the same function.





## Additional Examples

**4** If $f(x) = x^2 - 2x - 8$, find each function value.

**a.** $f(3)$ $-5$

**b.** $f(-3d)$ $9d^2 + 6d - 8$

**c.** $f(2a - 1)$ $4a^2 - 8a - 5$

**5** State the domain of each function.

**a.** $g(x) = \sqrt{4x - 1}$ $\left[\dfrac{1}{4}, \infty\right)$

**b.** $h(t) = \dfrac{3t^2}{t^2 - 1}$ $\{t \mid t \neq -1, t \neq 1, t \in \mathbb{R}\}$

**c.** $f(x) = \dfrac{x - 5}{\sqrt{2x - 3}}$ $\left\{x \,\middle|\, x > \dfrac{3}{2}, x \in \mathbb{R}\right\}$

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## Additional Example

**6** **FINANCE** Realtors in a metropolitan area studied the average home price per square meter as a function of total square meterage. Their evaluation yielded the following piecewise-defined function. Find the average price per square meter for a home with the given square meterage.

$$p(a) = \begin{cases} \dfrac{a - 1000}{40} + 75 & \text{if } 1000 \le a < 2600 \\ \dfrac{-(a - 2600)}{100} + 110 & \text{if } 2600 \le a < 4000 \\ \dfrac{a - 4000}{25} + 98 & \text{if } a \ge 4000 \end{cases}$$

**a.** 1400 $m^2$ AED 85 /$m^2$

**b.** 3200 $m^2$ AED 104 /$m^2$

**c.** $h(x) = \dfrac{1}{\sqrt{x^2 - 9}}$

This function is defined only when $x^2 - 9 > 0$. Therefore, the domain of $h(x)$ is $(-\infty, -3) \cup (3, \infty)$.

### Guided Practice

**State the domain of each function.** **5B. $(-\infty, -2] \cup [2, \infty)$**

**5A.** $f(x) = \dfrac{5x - 2}{x^2 + 7x + 12}$ **$(-\infty, -4) \cup (-4, -3) \cup (-3, \infty)$**

**5B.** $h(a) = \sqrt{a^2 - 4}$

**5C.** $g(x) = \dfrac{8x}{\sqrt{2x + 6}}$ **$(-3, \infty)$**

A function that is defined using two or more equations for different intervals of the domain is called a **piecewise-defined function**.

### Real-World Example 6 Evaluate a Piecewise-Defined Function

**HEIGHT** **The average maximum height of children in centimeters as a function of their parents' maximum heights in centimeters can be modeled by the following piecewise function. Find the average maximum heights of children whose parents have the given maximum heights. Use $h(x)$, where $x$ is the independent variable representing the parents' height and $h(x)$ is the dependent variable representing the child's height.**

$$h(x) = \begin{cases} 1.6x - 105 & \text{if } 160 < x < 167 \\ 3x - 335 & \text{if } 167 \le x \le 172 \\ 2x - 167 & \text{if } x > 172 \end{cases}$$

**a.** $h(170)$

Because 170 is between 167 and 172, use $h(x) = 3x - 335$ to find $h(170)$.

| | |
|---|---|
| $h(170) = 3x - 335$ | Function for $167 \le x \le 172$ |
| $= 3(170) - 335$ | Substitute 170 for $x$. |
| $= 510 - 335$ or $175$ | Simplify. |

According to this model, children whose parents have a maximum height of 170 cm will attain an average maximum height of 175 cm.

**b.** $h(182)$

Because 182 is greater than 172, use $h(x) = 2x - 167$.

| | |
|---|---|
| $h(182) = 2x - 167$ | Function for $x > 172$ |
| $= 2(182) - 167$ | Substitute 182 for $x$. |
| $= 364 - 167$ or $198$ | Simplify. |

According to this model, children whose parents have a maximum height of 182 cm will attain an average maximum height of 198 cm.

### Guided Practice

**6.** **SPEED** The speed $v$ of a vehicle in kilometers per hour can be represented by the following piecewise function when $t$ is the time in seconds. Find the speed of the vehicle at each indicated time.

$$v(t) = \begin{cases} 4t & \text{if } 0 \le t \le 15 \\ 60 & \text{if } 15 < t < 240 \\ -6t + 1500 & \text{if } 240 \le t \le 250 \end{cases}$$

**A.** $v(5)$ **20 km/h**  **B.** $v(15)$ **60 km/h**  **C.** $v(245)$ **30 km/h**

**Study Tip**

**Relevant Domain** A **relevant domain** is the part of a domain that is relevant to a model. Consider a function in which the output is a function of length. It is unreasonable to have a negative length, so the relevant domain is the set of numbers greater than or equal to 0.





## Additional Answers

1. $\{x \mid x > 50, x \in \mathbb{R}\}$; $(50, \infty)$
2. $\{x \mid x < -13, x \in \mathbb{R}\}$; $(-\infty, -13)$
3. $\{x \mid x \le -4, x \in \mathbb{R}\}$; $(-\infty, -4]$
4. $\{x \mid -4 \le x, x \in \mathbb{Z}\}$
5. $\{x \mid 8 < x < 99, x \in \mathbb{R}\}$; $(8, 99)$
6. $\{x \mid -31 < x \le 64, x \in \mathbb{R}\}$; $(-31, 64]$
7. $\{x \mid x < -19$ or $x > 21, x \in \mathbb{R}\}$; $(\infty, -19) \cup (21, \infty)$
8. $\{x \mid x < 0$ or $x \ge 100, x \in \mathbb{R}\}$; $(-\infty, 0) \cup [100, \infty)$
9. $\{x \mid 0.25n = x, n \ge -1, n \in \mathbb{Z}\}$
10. $\{x \mid x \le 61$ or $x \ge 67, x \in \mathbb{R}\}$; $(-\infty, 61] \cup [67, \infty)$

## Exercises

Write each set of numbers in set-builder and interval notation, if possible. (Examples 1 and 2) **1–14. See margin.**

1. $x > 50$
2. $x < -13$
3. $x \le -4$
4. $\{-4, -3, -2, -1, \ldots\}$
5. $8 < x < 99$
6. $-31 < x \le 64$
7. $x < -19$ or $x > 21$
8. $x < 0$ or $x \ge 100$
9. $\{-0.25, 0, 0.25, 0.50, \ldots\}$
10. $x \le 61$ or $x \ge 67$
11. $x \le -45$ or $x > 86$
12. all multiples of 8
13. all multiples of 5
14. $x \ge 32$

Determine whether each relation represents $y$ as a function of $x$. (Example 3)

15. The input value $x$ is a bank account number and the output value $y$ is the account balance. **function**

16. The input value $x$ is the year and the output value $y$ is the day of the week. **not a function**

17.

| $x$ | $y$ |
|---|---|
| −50 | 2.11 |
| −40 | 2.14 |
| −30 | 2.16 |
| −20 | 2.17 |
| −10 | 2.17 |
| 0 | 2.18 |

**function**

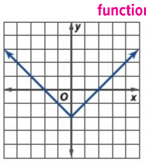
18.

| $x$ | $y$ |
|---|---|
| 0.01 | 423 |
| 0.04 | 449 |
| 0.04 | 451 |
| 0.07 | 466 |
| 0.08 | 478 |
| 0.09 | 482 |

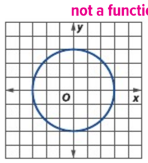
**not a function**

19. $\frac{1}{x} = y$ **function**
20. $x^2 = y + 2$ **function**
21. $3y + 4x = 11$ **function**
22. $4y^2 + 18 = 96x$ **not a function**
23. $\sqrt{48y} = x$ **function**
24. $\frac{x}{y} = y - 6$ **not a function**

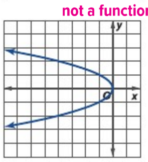
25. **function**



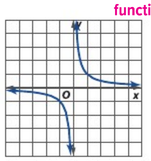
26. **not a function**



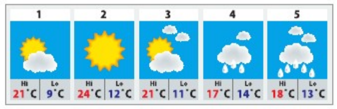
27. **not a function**



28. **function**



29. **METEOROLOGY** The five-day forecast for a city is shown. (Example 3)




a. Represent the relation between the day of the week and the estimated high temperature as a set of ordered pairs. **{(1, 21), (2, 24), (3, 21), (4, 17), (5, 18)}**

b. Is the estimated high temperature a function of the day of the week? the low temperature? Explain your reasoning. **Yes; there is exactly one estimated high temperature each day. Yes; there is exactly one low temperature each day.**

Find each function value. (Example 4) **32–35. See margin.**

30. $g(x) = 2x^2 + 18x - 14$
 a. $g(9)$ **310**
 b. $g(3x)$ **$18x^2 + 54x - 14$**
 c. $g(1 + 5m)$ **$50m^2 + 110m + 6$**

31. $h(y) = -3y^3 - 6y + 9$
 a. $h(4)$ **−207**
 b. $h(-2y)$ **$24y^3 + 12y + 9$**
 c. $h(5b + 3)$ **$-375b^3 - 675b^2 - 435b - 90$**

32. $f(t) = \frac{4t + 11}{3t^2 + 5t + 1}$
 a. $f(-6)$
 b. $f(4t)$
 c. $f(3 - 2a)$

33. $g(x) = \frac{3x^3}{x^2 + x - 4}$
 a. $g(-2)$
 b. $g(5x)$
 c. $g(8 - 4b)$

34. $h(x) = 16 - \frac{12}{2x + 3}$
 a. $h(-3)$
 b. $h(6x)$
 c. $h(10 - 2c)$

35. $f(x) = -7 + \frac{6x + 1}{x}$
 a. $f(5)$
 b. $f(-8x)$
 c. $f(6y + 4)$

36. $g(m) = 3 + \sqrt{m^2 - 4}$
 a. $g(-2)$ **3**
 b. $g(3m)$ **$3 + \sqrt{9m^2 - 4}$**
 c. $g(4m - 2)$ **$3 + 4\sqrt{m^2 - m}$**

37. $t(x) = 5\sqrt{6x^2}$
 a. $t(-4)$ **$20\sqrt{6}$**
 b. $t(2x)$ **$10|x|\sqrt{6}$**
 c. $t(7 + n)$ **$5|7 + n|\sqrt{6}$**

38. **DIGITAL AUDIO PLAYERS** The sales of digital audio players in millions of dirhams for a five-year period can be modeled using $f(t) = 24t^2 - 93t + 78$, where $t$ is the year. The actual sales data are shown in the table. (Example 4)

| Year | Sales (AED) |
|---|---|
| 1 | 1 million |
| 2 | 3 million |
| 3 | 14 million |
| 4 | 74 million |
| 5 | 219 million |

a. Find $f(1)$ and $f(5)$. **AED 9 million; AED 213 million**

b. Do you think that the model is more accurate for the earlier years or the later years? Explain your reasoning. **See margin.**





## 3 Practice

### Formative Assessment

Use Exercises 1–53 to check for understanding.

Then use the table below to customize your assignments for students.

### WatchOut!

**Common Error** If students need help with Exercises 12 and 13, write this sequence for Exercise 12: 8(0), 8(±1), 8(±2), 8(±3), ...

This will make it clearer that $n$ will be 0, ±1, ±2, ±3, ... , or all integers.

### Additional Answers

**11.** $\{x \mid x \le -45 \text{ or } x > 86, x \in \mathbb{R}\}$; $(-\infty, -45] \cup (86, \infty)$

**12.** $\{x \mid x = 8n, n \in \mathbb{Z}\}$

**13.** $\{x \mid x = 5n, n \in \mathbb{Z}\}$

**14.** $\{x \mid x \ge 32, x \in \mathbb{R}\}$; $[32, \infty)$

**32a.** $-\frac{13}{79}$

**32b.** $\frac{16t + 11}{48t^2 + 20t + 1}$

**32c.** $\frac{-8a + 23}{12a^2 - 46a + 43}$

**33a.** 12

**33b.** $\frac{375x^3}{25x^2 + 5x - 4}$

**33c.** $\frac{-48b^3 + 288b^2 - 576b + 384}{4b^2 - 17b + 17}$

**34a.** 20

**34b.** $16 - \frac{4}{4x + 1}$

**34c.** $16 - \frac{12}{23 - 4c}$

**35a.** −0.8

**35b.** $-1 - \frac{1}{8x}$

**35c.** $-7 + \frac{36y + 25}{6y + 4}$

**38b.** Sample answer: I think the model is closer for the later years, which have the higher sales numbers, because 213 is within 2% of 219 and 9 is 800% larger than 1.

## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–53, 80, 81, 83–110 | 1–53 odd, 107–110 | 2–52 even, 80, 81, 83–106 |
| OL On Level | 1–57 odd, 58–62, 63–73 odd, 75–77, 79–81, 83–110 | 1–53, 107–110 | 54–81, 83–106 |
| BL Beyond Level | 54–110 | | |

**WatchOut!**

**Common Error** For Exercises 39–46, remind students of two basic rules.

1. The denominator cannot equal zero.
2. There is no real square root of a negative number.

**Error Analysis** For Exercise 80, Ana's answer missed the numbers between −2 and 2. Remind students that only $x$-values that cause the denominator to equal 0 are excluded from the domain.

## Tips for New Teachers

**Relations and Functions** In Exercises 77 and 78, identifying coordinate pairs can be a quick way to determine whether a relation is a function. It is not always necessary to graph each relation.

## Additional Answers

**39.** $(-\infty, -4) \cup (-4, -1) \cup (-1, \infty)$

**40.** $(-\infty, -5) \cup (-5, 8) \cup (8, \infty)$

**41.** $(-\infty, \infty)$

**42.** $\left[-\sqrt{6}, \sqrt{6}\right]$

**43.** $(0.25, \infty)$

**44.** $(-\infty, -4) \cup (4, \infty)$

**45.** $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$

**46.** $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

**54.** Yes; a vertical line would not pass through the graph more than once.

**55.** No; a vertical line would pass through infinitely many points.

**56.** Yes; a vertical line would not pass through the graph more than once.

**57.** No; the $y$-axis is a vertical line that passes through two points of the graph, (0, 0) and (0, −4).

**58a.** $D(t) = \begin{cases} 4t \text{ if } 0 \le t \le 0.6 \\ 20t - 9.6 \text{ if } 0.6 < t \le 6.2 \\ 6t + 77.2 \text{ if } 6.2 < t \le 10.6 \end{cases}$

**59.** $\{x \mid x = 4n + 1792, n \in \mathbb{W}\}$; Sample answer: Because presidential elections are held every 4 years, and do not have a finite end, it is impractical to display the set in interval notation. If set-builder notation is used, the interval can be taken into account and a finite interval is not necessary.

**State the domain of each function.** (Example 5) **39–46. See margin.**

**39.** $f(x) = \dfrac{8x + 12}{x^2 + 5x + 4}$

**40.** $g(x) = \dfrac{x + 1}{x^2 - 3x - 40}$

**41.** $g(a) = \sqrt{1 + a^2}$

**42.** $h(x) = \sqrt{6 - x^2}$

**43.** $f(a) = \dfrac{5a}{\sqrt{4a - 1}}$

**44.** $g(x) = \dfrac{3}{\sqrt{x^2 - 16}}$

**45.** $f(x) = \dfrac{2}{x} + \dfrac{4}{x + 1}$

**46.** $g(x) = \dfrac{6}{x + 3} + \dfrac{2}{x - 4}$

**47.** **PHYSICS** The period $T$ of a pendulum is the time for one cycle and can be calculated using the formula $T = 2\pi\sqrt{\frac{\ell}{9.8}}$, where $\ell$ is the length of the pendulum and 9.8 is the acceleration due to gravity in meters per second squared. Is this formula a function of $\ell$? If so, determine the domain. If not, explain why not. (Example 5) **Yes; sample answer: Because length must be positive, the domain of the function is $(0, \infty)$.**


**Find $f(-5)$ and $f(12)$ for each piecewise function.** (Example 6)

**48.** $f(x) = \begin{cases} -4x + 3 & \text{if } x < 3 \\ -x^3 & \text{if } 3 \le x \le 8 \\ 3x^2 + 1 & \text{if } x > 8 \end{cases}$ **23; 433**

**49.** $f(x) = \begin{cases} -5x^2 & \text{if } x < -6 \\ x^2 + x + 1 & \text{if } -6 \le x \le 12 \\ 0.5x^3 - 4 & \text{if } x > 12 \end{cases}$ **21; 157**

**50.** $f(x) = \begin{cases} 2x^2 + 6x + 4 & \text{if } x < -4 \\ 6 - x^2 & \text{if } -4 \le x < 12 \\ 14 & \text{if } x \ge 12 \end{cases}$ **24; 14**

**51.** $f(x) = \begin{cases} -15 & \text{if } x < -5 \\ \sqrt{x + 6} & \text{if } -5 \le x \le 10 \\ \frac{2}{x} + 8 & \text{if } x > 10 \end{cases}$ **1; $8\frac{1}{6}$**

**52.** **INCOME TAX** Federal income tax for a person filing single in the United States in a recent year can be modeled using the following function, where $x$ represents income and $T(x)$ represents total tax. (Example 6)

$$T(x) = \begin{cases} 0.10x & \text{if } 0 \le x \le 7285 \\ 782.5 + 0.15x & \text{if } 7285 < x \le 31{,}850 \\ 4386.25 + 0.25x & \text{if } 31{,}850 < x \le 77{,}100 \end{cases}$$

a. Find $T(7000)$, $T(10{,}000)$, and $T(50{,}000)$. **AED 700, AED 2282.5, AED 16,886.25**

b. If a person's annual income were AED 7285, what would his or her income tax be? **AED 728.50**

**53.** **PUBLIC TRANSPORTATION** The nationwide use of public transportation can be modeled using the following function. The year 2012 is represented by $t = 0$, and $P(t)$ represents passenger trips in millions. (Example 6)

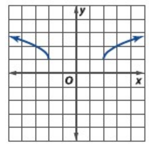
$$P(t) = \begin{cases} 0.35t + 7.6 & \text{if } 0 \le t \le 5 \\ 0.04t^2 - 0.6t + 11.6 & \text{if } 5 < t \le 10 \end{cases}$$

a. Approximately how many passenger trips were there in 2016? in 2020? **9.0 million; 9.36 million**

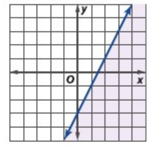
b. State the domain of the function. **integral values inside the interval [0, 10]**

**Use the vertical line test to determine whether each graph represents a function. Write *yes* or *no*. Explain your reasoning.** **54–57. See margin.**

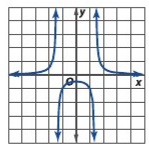
**54.**



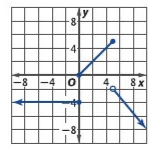
**55.**



**56.**



**57.**



**58.** **TRIATHLON** In a triathlon, athletes swim 2.4 km, then bike 112 km, and finally run 26.2 km. Mahmoud's average rates for each leg of a triathlon are shown in the table.

| Leg | Rate |
|---|---|
| swim | 4 km/h |
| bike | 20 km/h |
| run | 6 km/h |

a. Write a piecewise function to describe the distance $D$ that Mahmoud has traveled in terms of time $t$. Round $t$ to the nearest tenth, if necessary. **See margin.**

b. State the domain of the function. **[0, 10.6]**

**59.** **ELECTIONS** Describe the set of US presidential election years beginning in 1792 in interval notation or in set-builder notation. Explain your reasoning. **See margin.**

**60.** **CONCESSIONS** The number of students working the concession stands at a football game can be represented by $f(x) = \frac{x}{50}$, where $x$ is the number of tickets sold. Describe the relevant domain of the function. **The domain of the function is the set of whole numbers from 0 to the capacity of the stadium.**

**61. ATTENDANCE** The Chicago Cubs baseball franchise has been in existence since 1876. The total season attendance for its home games can be modeled by $f(x) = 21{,}870x - 40{,}962{,}679$, where $x$ represents the year. Describe the relevant domain of the function. **$D = \{x \mid x \geq 1876, x \in \mathbb{W}\}$**

**62. ACCOUNTING** A business' assets, such as equipment, wear out or depreciate over time. One way to calculate depreciation is the straight-line method, using the value of the estimated life of the asset. Suppose $v(t) = 10{,}440 - 290t$ describes the value $v(t)$ of a copy machine after $t$ months. Describe the relevant domain of the function. **$D = \{t \mid 0 \leq t \leq 36, t \in \mathbb{R}\}$**

**Find $f(a)$, $f(a + h)$, and $\frac{f(a+h) - f(a)}{h}$ if $h \neq 0$.** **63–74. See margin.**

**63.** $f(x) = -5$

**64.** $f(x) = \sqrt{x}$

**65.** $f(x) = \frac{1}{x + 4}$

**66.** $f(x) = \frac{2}{5 - x}$

**67.** $f(x) = x^2 - 6x + 8$

**68.** $f(x) = -\frac{1}{4}x + 6$

**69.** $f(x) = -x^5$

**70.** $f(x) = x^3 + 9$

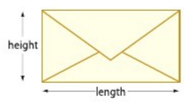
**71.** $f(x) = 7x - 3$

**72.** $f(x) = 5x^2$

**73.** $f(x) = x^3$

**74.** $f(x) = 11$

**75. MAIL** A Postal Service requires that envelopes have an aspect ratio (length divided by height) of 1.3 to 2.5, inclusive. The minimum allowable length is 12.5 cm and the maximum allowable length is 28.5 cm.

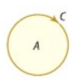



**a.** Write the area of the envelope $A$ as a function of length $\ell$ if the aspect ratio is 1.8. State the domain of the function. **$A(\ell) = \frac{\ell^2}{1.8}$; [12.5, 28.5]**

**b.** Write the area of the envelope $A$ as a function of height $h$ if the aspect ratio is 2.1. State the domain of the function. **$A(h) = 2.1h^2$; [6, 13.5]**

**c.** Find the area of an envelope with the maximum height at the maximum aspect ratio. **330.6 cm$^2$**

**76. GEOMETRY** Consider the circle below with area $A$ and circumference $C$.




**a.** Represent the area of the circle as a function of its circumference. **$A = \frac{C^2}{4\pi}$**

**b.** Find $A(0.5)$ and $A(4)$. **0.02, 1.27**

**c.** What do you notice about the area as the circumference increases? **As the circumference increases, the area also increases.**

**Determine whether each equation is a function of $x$. Explain.**

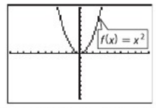
**77.** $x = |y|$

**78.** $x = y^3$

**77–78. See Chapter 11 Answer Appendix.**

**79. MULTIPLE REPRESENTATIONS** In this problem, you will investigate the range of a function.

**a. GRAPHICAL** Use a graphing calculator to graph $f(x) = x^n$ for whole-number values of $n$ from 1 to 6, inclusive.




[−10, 10] scl: 1 by [−10, 10] scl: 1

**a–c. See Chapter 11 Answer Appendix.**

**b. TABULAR** Predict the range of each function based on the graph, and tabulate each value of $n$ and the corresponding range.

**c. VERBAL** Make a conjecture about the range of $f(x)$ when $n$ is even.

**d. VERBAL** Make a conjecture about the range of $f(x)$ when $n$ is odd. **Sample answer: When $n$ is odd in $f(x) = x^n$, the range is $(-\infty, \infty)$.**

## H.O.T. Problems Use Higher-Order Thinking Skills

**80. ERROR ANALYSIS** Ahmed and Tarek are evaluating $f(x) = \frac{2}{x^2 - 4}$. Ahmed thinks that the domain of the function is $(-\infty, -2) \cup (1, 1) \cup (2, \infty)$. Tarek thinks that the domain is $\{x \mid x \neq -2, x \neq 2, x \in \mathbb{R}\}$. Is either of them correct? Explain. **See Chapter 11 Answer Appendix.**

**81. WRITING IN MATH** Write the domain of $f(x) = \frac{1}{(x + 3)(x + 1)(x - 5)}$ in interval notation and in set-builder notation. Which notation do you prefer? Explain. **See Chapter 11 Answer Appendix.**

**82. CHALLENGE** $G(x)$ is a function for which $G(1) = 1$, $G(2) = 2$, $G(3) = 3$, and $G(x + 1) = \frac{G(x - 2)\,G(x - 1) + 1}{G(x)}$ for $x \geq 3$. Find $G(6)$. **$\frac{4}{7}$**

**REASONING Determine whether each statement is *true* or *false* given a function from set $X$ to set $Y$. If a statement is false, rewrite it to make a true statement.**

**83.** Every element in $X$ must be matched with only one element in $Y$. **true**

**84.** Every element in $Y$ must be matched with an element in $X$. **See Chapter 11 Answer Appendix.**

**85.** Two or more elements in $X$ may not be matched with the same element in $Y$. **See Chapter 11 Answer Appendix.**

**86.** Two or more elements in $Y$ may not be matched with the same element in $X$. **true**

**WRITING IN MATH Explain how you can identify a function described as each of the following.**

**87.** a verbal description of inputs and outputs

**88.** a set of ordered pairs

**89.** a table of values

**90.** a graph

**91.** an equation

**87–91. See Chapter 11 Answer Appendix.**





## Additional Answers

**63.** $-5; -5; 0$

**64.** $\sqrt{a}; \sqrt{a + h}; \frac{\sqrt{a + h} - \sqrt{a}}{h}$

**65.** $\frac{1}{a + 4}; \frac{1}{a + h + 4}; \frac{-1}{a^2 + ah + 8a + 4h + 16}$

**66.** $\frac{2}{5 - a}; \frac{2}{5 - a - h}; \frac{2}{a^2 - 10a + ah - 5h + 25}$

**67.** $a^2 - 6a + 8; a^2 + 2ah + h^2 - 6a - 6h + 8; 2a - 6 + h$

**68.** $-\frac{1}{4}a + 6; \frac{-a - h}{4} + 6; -\frac{1}{4}$

**69.** $-a^5; -a^5 - 5a^4h - 10a^3h^2 - 10a^2h^3 - 5ah^4 - h^5; -5a^4 - 10a^3h - 10a^2h^2 - 5ah^3 - h^4$

**70.** $a^3 + 9; a^3 + 3a^2h + 3ah^2 + h^3 + 9; 3a^2 + 3ah + h^2$

**71.** $7a - 3; 7a + 7h - 3; 7$

**72.** $5a^2; 5a^2 + 10ah + 5h^2; 10a + 5h$

**73.** $a^3; a^3 + 3a^2h + h^3; 3a^2 + 3ah + h^2$

**74.** 11; 11; 0

# 4 Assess

**Ticket Out the Door** Given $f(x) = \frac{-4x}{\sqrt{x^2 - 1}}$, evaluate $f(3)$. $-3\sqrt{2}$



## Spiral Review

**Find the standard deviation of each population of data.**

**92.** {200, 476, 721, 579, 152, 158} **223.14**

**93.** {5.7, 5.7, 5.6, 5.5, 5.3, 4.9, 4.4, 4.0, 4.0, 3.8} **0.73**

**94.** {369, 398, 381, 392, 406, 413, 376, 454, 420, 385, 402, 446} **25.31**

**95. BASEBALL** How many different 9-player teams can be made if there are 3 players who can only play catcher, 4 players who can only play first base, 6 players who can only pitch, and 14 players who can play in any of the remaining 6 positions? **216,216**

**Find the values for $x$ and $y$ that make each matrix equation true.**

**96.** $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 4x - 3 \\ y - 2 \end{bmatrix}$ $\left(1\frac{2}{3}, 3\frac{2}{3}\right)$

**97.** $\begin{bmatrix} 3y \\ 10 \end{bmatrix} = \begin{bmatrix} 27 + 6x \\ 5y \end{bmatrix}$ **(−3.5, 2)**

**98.** $[9 \quad 11] = [3x + 3y \quad 2x + 1]$ **(5, −2)**

**Use any method to solve the system of equations. State whether the system is *consistent*, *dependent*, *independent*, or *inconsistent*.**

**99.** $2x + 3y = 36$
$4x + 2y = 48$ **(9, 6); consistent and independent**

**100.** $5x + y = 25$
$10x + 2y = 50$ **infinitely many solutions; consistent and dependent**

**101.** $7x + 8y = 30$
$7x + 16y = 46$ **(2, 2); consistent and independent**

**102. BUSINESS** A used book store sells 1400 paperback books per week at AED 9 per book. The owner estimates that he will sell 100 fewer books for each AED 1 increase in price. What price will maximize the income of the store? **AED 11.52**

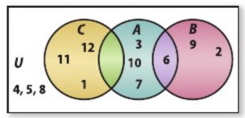
**Use the Venn diagram to find each of the following.**

**103.** $A'$ **{1, 2, 4, 5, 8, 9, 11, 12}**

**104.** $A \cup B$ **{2, 3, 6, 7, 9, 10}**

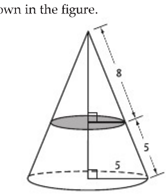
**105.** $B \cap C$ **∅**

**106.** $A \cap B$ **{6}**




## Skills Review for Standardized Tests

**107. SAT/ACT** A circular cone with a base of radius 5 has been cut as shown in the figure.




What is the height of the smaller top cone? **B**

**A** $\frac{8}{13}$ **B** $\frac{96}{13}$ **C** $\frac{96}{12}$ **D** $\frac{96}{5}$ **E** $\frac{104}{5}$

**108. REVIEW** Which function is linear? **G**

**F** $f(x) = x^2$

**G** $g(x) = 2.7$

**H** $f(x) = \sqrt{9 - x^2}$

**J** $g(x) = \sqrt{x - 1}$

**109.** Omar is flying from Denver to Dallas for a convention. He can park his car in the Denver airport long-term lot or in the nearby shuttle parking facility. The long-term lot costs AED 4 per hour or any fraction thereof with a maximum charge of AED 24 per day. In the shuttle facility, he has to pay AED 16 for each day or part of a day. Which parking lot is less expensive if Omar returns after 2 days and 3 hours? **A**

**A** shuttle facility

**B** airport lot

**C** They will both cost the same.

**D** cannot be determined with the information given

**110. REVIEW** Given $y = 2.24x + 16.45$, which statement best describes the effect of moving the graph down two units? **H**

**F** The $y$-intercept increases.

**G** The $x$-intercept remains the same.

**H** The $x$-intercept increases.

**J** The $y$-intercept remains the same.





## Differentiated Instruction BL

**Extension** Have students work in small groups. Ask each group to find two examples of functions that have the domain $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$. Sample answers: $f(x) = \frac{1}{x^2 + 2x - 3}$, $g(x) = \frac{x - 4}{x^2 + 2x - 3}$

# LESSON 11-2 Analyzing Graphs of Functions and Relations

**Then**

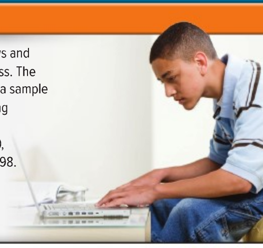
- You identified functions. (Lesson 11-1)

**Now**

1. Use graphs of functions to estimate function values and find domains, ranges, $y$-intercepts, and zeros of functions.
2. Explore symmetries of graphs, and identify even and odd functions.

**Why?**

- With more people turning to the Internet for news and entertainment, Internet advertising is big business. The total revenue $R$ in millions of dirhams earned by a sample of worldwide companies from Internet advertising from 1999 to 2008 can be approximated by $R(t) = 17.7t^3 - 269t^2 + 1458t - 910$, $1 \le t \le 10$, where $t$ represents the number of years since 1998. Graphs of functions like this can help you visualize relationships between real-world quantities.

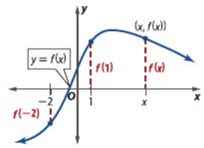


**NewVocabulary**

- zeros
- roots
- line symmetry
- point symmetry
- even function
- odd function

## 1 Analyzing Function Graphs

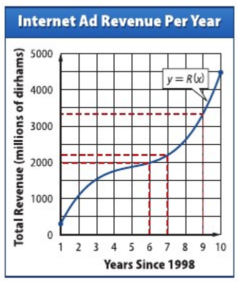
The graph of a function $f$ is the set of ordered pairs $(x, f(x))$ such that $x$ is in the domain of $f$. In other words, the graph of $f$ is the graph of the equation $y = f(x)$. So, the value of the function is the directed distance $y$ of the graph from the point $x$ on the $x$-axis as shown.




You can use a graph to estimate function values.

### Real-World Example 1 Estimate Function Values

**INTERNET** **Consider the graph of function $R$ shown.**




**a. Use the graph to estimate total Internet advertising revenue in 2007. Confirm the estimate algebraically.**

The year 2007 is 9 years after 1998. The function value at $x = 9$ appears to be about AED 3300 million, so the total Internet advertising revenue in 2007 was about AED 3.3 billion.

To confirm this estimate algebraically, find $f(9)$.

$f(9) = 17.7(9)^3 - 269(9)^2 + 1458(9) - 910$

$\approx 3326.3$ million or 3.326 billion

Therefore, the graphical estimate of AED 3.3 billion is reasonable.

**b. Use the graph to estimate the year in which total Internet advertising revenue reached AED 2 billion. Confirm the estimate algebraically.**

The value of the function appears to reach AED 2 billion or AED 2000 million for $x$-values between 6 and 7. So, the total revenue was nearly AED 2 billion in 1998 + 6 or 2004 but had exceeded AED 2 billion by the end of 1998 + 7 or 2005.

To confirm algebraically, find $f(6)$ and $f(7)$.

$f(6) = 17.7(6)^3 - 269(6)^2 + 1458(6) - 910$ or about 1977 million

$f(7) = 17.7(7)^3 - 269(7)^2 + 1458(7) - 910$ or about 2186 million

In billions, $f(6) \approx 1.977$ billion and $f(7) \approx 2.186$ billion. Therefore, the graphical estimate that total Internet advertising revenue reached AED 2 billion in 2005 is reasonable.





## 1 Focus

**VerticalAlignment**

**Before Lesson 11-2** Identify functions

**Lesson 11-2** Use graphs of functions to estimate function values and find domains, ranges, $y$-intercepts, and zeros of functions.
Explore symmetries of graphs, and identify even and odd functions.

**After Lesson 11-2** Explore continuity, end behavior, and limits.

## 2 Teach

### Scaffolding Questions

Have students read the **Why?** section of the lesson.

**Ask:**

- The linear graph of a function for net profit/loss given $x$ units sold has an $x$-intercept of 200. What does this mean? After 200 units are sold, there will be a profit.

*(continued on the next page)*

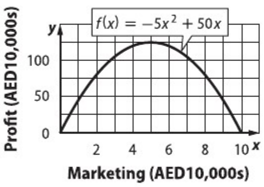
- A graph of marathon times given resting time prior to the race decreases and then increases with a minimum at two days. If marathon times are the dependent variable and resting time is the independent variable, what does this mean? The fastest marathon time occurs when the resting time is two days. Resting times greater than or less than two days increases times.

### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

### Additional Example

**1 ADVERTISING** The function $f(x) = -5x^2 + 50x$ approximates the profit at a toy company, where $x$ is marketing costs and $f(x)$ is profit. Both costs and profits are measured in tens of thousands of dirhams.

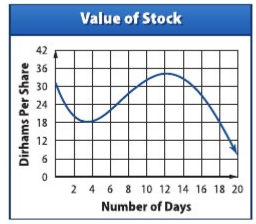



**a.** Use the graph to estimate the profit when marketing costs are AED 30,000. Confirm your estimate algebraically.
AED 1,050,000

**b.** Use the graph to estimate marketing costs when the profit is AED 1,250,000. Confirm your estimate algebraically. AED 50,000

**Guided Practice**

**1. STOCKS** An investor assessed the average daily value of a share of a certain stock over a 20-day period. The value of the stock can be approximated by $v(d) = 0.002d^4 - 0.11d^3 + 1.77d^2 - 8.6d + 31$, $0 \le d \le 20$, where $d$ represents the day of the assessment.



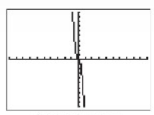

**A.** Use the graph to estimate the value of the stock on the 10th day. Confirm your estimate algebraically. **AED 32**

**B.** Use the graph to estimate the days during which the stock was valued at AED 30 per share. Confirm your estimate algebraically.
**day 0, between the 9th and 10th, and between the 15th and 16th**

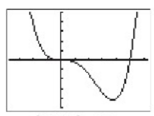
You can also use a graph to find the domain and range of a function. Unless the graph of a function is bounded on the left by a circle or a dot, you can assume that the function extends beyond the edges of the graph.

**Technology Tip**

**Choosing an Appropriate Window** The viewing window of a graph is a picture of the graph for a specific domain and range. This may not represent the entire graph. Notice the difference in the graphs of $f(x) = x^4 - 20x^3$ shown below.



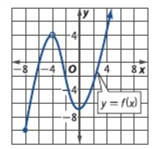
[−10, 10] scl: 1 by
[−10, 10] scl: 1



[−15, 25] scl: 4 by
[−20,000, 20,000] scl: 4000

**Example 2** Find Domain and Range

**Use the graph of $f$ to find the domain and range of the function.**




**Domain**

- The dot at $(-8, -10)$ indicates that the domain of $f$ starts at and includes $-8$.
- The circle at $(-4, 4)$ indicates that $-4$ is not part of the domain.
- The arrow on the right side indicates that the graph will continue without bound.

The domain of $f$ is $[-8, -4) \cup (-4, \infty)$. In set-builder notation, the domain is $\{x \mid -8 \le x, x \ne -4, x \in \mathbb{R}\}$.

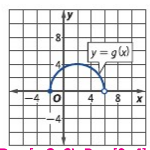
**Range**

The graph does not extend below $f(-8)$ or $-10$, but $f(x)$ increases without bound for greater and greater values of $x$. So, the range of $f$ is $[-10, \infty)$.

**Guided Practice**

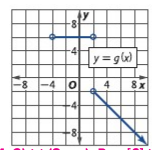
**Use the graph of $g$ to find the domain and range of each function.**

**2A.**




**D = [−2, 6), R = [0, 4]**

**2B.**




**D = (−4, 2) ∪ (2, ∞), R = [6] ∪ (−2, −∞)**

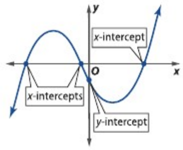




### 1 Analyzing Function Graphs

**Example 1** shows how to use a graph to estimate function values. **Example 2** shows how to use a graph to find the domain and range of a function. **Example 3** shows how to find $y$-intercepts of functions. **Example 4** shows how to find zeros of a function.

A point where a graph intersects or meets the $x$- or $y$-axis is called an intercept. An $x$-intercept of a graph occurs where $y = 0$. A $y$-intercept of a graph occurs where $x = 0$. The graph of a function can have 0, 1, or more $x$-intercepts, but at most one $y$-intercept.

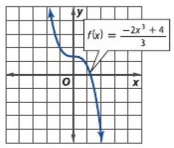



To find the $y$-intercept of a graph of a function $f$ algebraically, find $f(0)$.

**Example 3** Find $y$-Intercepts

**Use the graph of each function to approximate its $y$-intercept. Then find the $y$-intercept algebraically.**

**a.**




**Estimate Graphically**

It appears that $f(x)$ intersects the $y$-axis at approximately $\left(0, 1\frac{1}{3}\right)$, so the $y$-intercept is about $1\frac{1}{3}$.

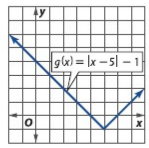
**Solve Algebraically**

Find $f(0)$.

$f(0) = \frac{-2(0)^3 + 4}{3}$ or $\frac{4}{3}$

The $y$-intercept is $\frac{4}{3}$ or $1\frac{1}{3}$.

**b.**




**Estimate Graphically**

It appears that $g(x)$ intersects the $y$-axis at $(0, 4)$, so the $y$-intercept is 4.

**Solve Algebraically**

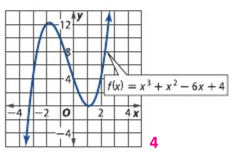
Find $g(0)$.

$g(0) = |0 - 5| - 1$ or 4

The $y$-intercept is 4.

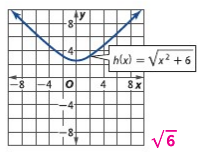
**Guided Practice**

**3A.**




4

**3B.**




$\sqrt{6}$

The $x$-intercepts of the graph of a function are also called the **zeros** of a function. The solutions of the corresponding equation are called the **roots** of the equation. To find the zeros of a function $f$, set the function equal to 0 and solve for the independent variable.

**StudyTip**

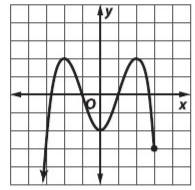
**Labeling Axis on Graphs** When you label an axis on the graph, the variable letter for the domain is on the $x$-axis and the variable letter for the range is on the $y$-axis. Throughout this book, there are many different variables used for both the domain and range. For consistency, the horizontal axis is always $x$ and the vertical axis is always $y$.





## Additional Examples

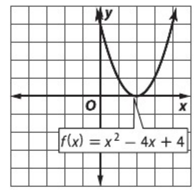
**2** Use the graph of $f$ to find the domain and range of the function.




D = $(-\infty, 3]$, R = $(-\infty, 2]$

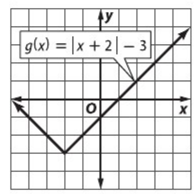
**3** Use the graph of each function to approximate its $y$-intercept. Then find the $y$-intercept algebraically.

**a.**




4

**b.**




−1

## Teach with Tech

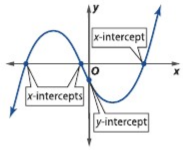
**Graphing Technology** Use a tracing application that allows students to view coordinates as the mouse is moved over the graph. This technology provides students with immediate feedback for their estimates of values.

## Focus on Mathematical Content

**Representing Functions** Graphical and algebraic representations of functions convey a large amount of information about the relationship between two values.

- Graphs allow easy identification of maximums, minimums, zeros, and $y$-intercepts.
- The equation of a function can yield exact values.

A point where a graph intersects or meets the $x$- or $y$-axis is called an intercept. An $x$-intercept of a graph occurs where $y = 0$. A $y$-intercept of a graph occurs where $x = 0$. The graph of a function can have 0, 1, or more $x$-intercepts, but at most one $y$-intercept.




To find the $y$-intercept of a graph of a function $f$ algebraically, find $f(0)$.

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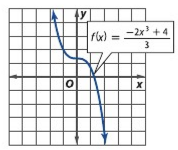
### Example 3 Find $y$-Intercepts

**Use the graph of each function to approximate its $y$-intercept. Then find the $y$-intercept algebraically.**

**a.**




**Estimate Graphically**

It appears that $f(x)$ intersects the $y$-axis at approximately $\left(0, 1\frac{1}{3}\right)$, so the $y$-intercept is about $1\frac{1}{3}$.

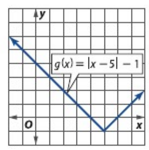
**Solve Algebraically**

Find $f(0)$.

$f(0) = \frac{-2(0)^3 + 4}{3}$ or $\frac{4}{3}$

The $y$-intercept is $\frac{4}{3}$ or $1\frac{1}{3}$.

**b.**




**Estimate Graphically**

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**Solve Algebraically**

Find $g(0)$.

$g(0) = |0 - 5| - 1$ or 4

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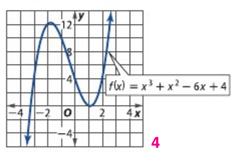
**GuidedPractice**

**3A.**

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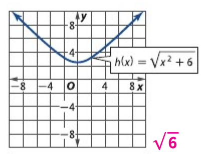
**3B.**




$\sqrt{6}$

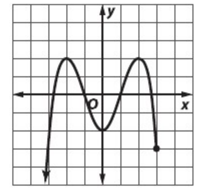
The $x$-intercepts of the graph of a function are also called the **zeros** of a function. The solutions of the corresponding equation are called the **roots** of the equation. To find the zeros of a function $f$, set the function equal to 0 and solve for the independent variable.





## Additional Examples

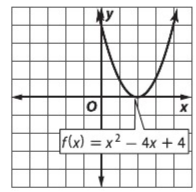
**2** Use the graph of $f$ to find the domain and range of the function.




D = $(-\infty, 3]$, R = $(-\infty, 2]$

**3** Use the graph of each function to approximate its $y$-intercept. Then find the $y$-intercept algebraically.

**a.**

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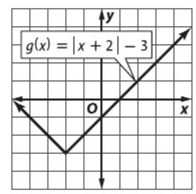
**b.**




−1

### Teach with Tech

**Graphing Technology** Use a tracing application that allows students to view coordinates as the mouse is moved over the graph. This technology provides students with immediate feedback for their estimates of values.

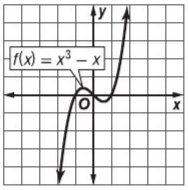
### Focus on Mathematical Content

**Representing Functions** Graphical and algebraic representations of functions convey a large amount of information about the relationship between two values.

- Graphs allow easy identification of maximums, minimums, zeros, and $y$-intercepts.
- The equation of a function can yield exact values.

## Additional Example

**4** Use the graph of $f(x) = x^3 - x$ to approximate its zeros. Then find its zeros algebraically.
$-1, 0, 1$




### Tips for New Teachers

**Finding Values Graphically** When using a graph to find function values, students should try using a straightedge that extends to each axis. This can be easier and more accurate.

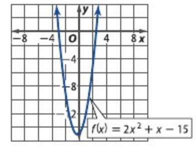
### 2 Symmetry of Graphs

**Example 5** shows how to test graphs of functions for line and point symmetry. **Example 6** shows how to determine whether a function is even or odd.

---

**Example 4** Find Zeros

**Use the graph of $f(x) = 2x^2 + x - 15$ to approximate its zero(s). Then find its zero(s) algebraically.**




**Estimate Graphically**

The $x$-intercepts appear to be at about $-3$ and $2.5$.

**Solve Algebraically**

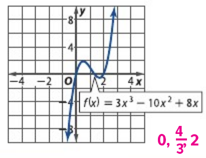
| | |
|---|---|
| $2x^2 + x - 15 = 0$ | Let $f(x) = 0$. |
| $(2x - 5)(x + 3) = 0$ | Factor. |
| $2x - 5 = 0$ or $x + 3 = 0$ | Zero Product Property |
| $x = 2.5$ $\quad x = -3$ | Solve for $x$. |

The zeros of $f$ are $-3$ and $2.5$.

**Guided Practice**

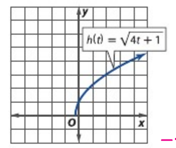
**Use the graph of each function to approximate its zero(s). Then find its zero(s) algebraically.**

**4A.**




$0, \frac{4}{3}, 2$

**4B.**




$-\frac{1}{4}$

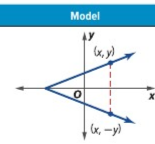
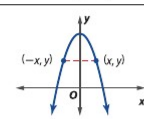
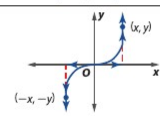
### 2 Symmetry of Graphs

Graphs of relations can have two different types of symmetry. Graphs with **line symmetry** can be folded along a line so that the two halves match exactly. Graphs that have **point symmetry** can be rotated 180° with respect to a point and appear unchanged. The three most common types of symmetry are shown below.

**StudyTip**

**Symmetry, Relations, and Functions** There are numerous *relations* that have $x$-axis, $y$-axis, and origin symmetry. However, the only *function* that has all three types of symmetry is the zero function, $f(x) = 0$.

**KeyConcept** Tests for Symmetry

| Graphical Test | Model | Algebraic Test |
|---|---|---|
| The graph of a relation is *symmetric with respect to the x-axis* if and only if for every point $(x, y)$ on the graph, the point $(x, -y)$ is also on the graph. |  | Replacing $y$ with $-y$ produces an equivalent equation. |
| The graph of a relation is *symmetric with respect to the y-axis* if and only if for every point $(x, y)$ on the graph, the point $(-x, y)$ is also on the graph. |  | Replacing $x$ with $-x$ produces an equivalent equation. |
| The graph of a relation is *symmetric with respect to the origin* if and only if for every point $(x, y)$ on the graph, the point $(-x, -y)$ is also on the graph. |  | Replacing $x$ with $-x$ and $y$ with $-y$ produces an equivalent equation. |





## Differentiated Instruction AL OL

**Visual Learners** Have students find independent and dependent variables of interest to them. Have them describe the variables and determine a realistic domain and range for the function. For example, a domain with negative numbers may make sense for temperature but not for time spent playing a game. Then have students graph their functions.

712 | Lesson 11-2 | Analyzing Graphs of Functions and Relations

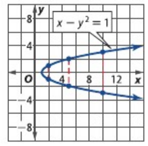
**StudyTip**

**Symmetry** It is possible for a graph to exhibit more than one type of symmetry.

## Example 5 Test for Symmetry

**Use the graph of each equation to test for symmetry with respect to the $x$-axis, $y$-axis, and the origin. Support the answer numerically. Then confirm algebraically.**

**a.** $x - y^2 = 1$




**Analyze Graphically**

The graph appears to be symmetric with respect to the $x$-axis because for every point $(x, y)$ on the graph, there is a point $(x, -y)$.

**Support Numerically**

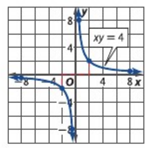
A table of values supports this conjecture.

| $x$ | 2 | 2 | 5 | 5 | 10 | 10 |
|---|---|---|---|---|---|---|
| $y$ | 1 | −1 | 2 | −2 | 3 | −3 |
| $(x, y)$ | (2, 1) | (2, −1) | (5, 2) | (5, −2) | (10, 3) | (10, −3) |

**Confirm Algebraically**

Because $x - (-y)^2 = 1$ is equivalent to $x - y^2 = 1$, the graph is symmetric with respect to the $x$-axis.

**b.** $xy = 4$




**Analyze Graphically**

The graph appears to be symmetric with respect to the origin because for every point $(x, y)$ on the graph, there is a point $(-x, -y)$.

**Support Numerically**

A table of values supports this conjecture.

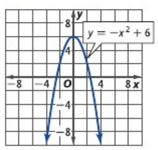
| $x$ | −8 | −2 | −0.5 | 0.5 | 2 | 8 |
|---|---|---|---|---|---|---|
| $y$ | −0.5 | −2 | −8 | 8 | 2 | 0.5 |
| $(x, y)$ | (−8, −0.5) | (−2, −2) | (−0.5, −8) | (0.5, 8) | (2, 2) | (8, 0.5) |

**Confirm Algebraically**

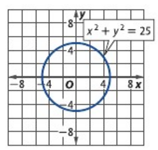
Because $(-x)(-y) = 4$ is equivalent to $xy = 4$, the graph is symmetric with respect to the origin.

### GuidedPractice

**5A.**




**5B.**




**5A.** Because $y = -(-x)^2 + 6$ is equivalent to $y = -x^2 + 6$, the graph is symmetric with respect to the $y$-axis.

**5B.** Because $x^2 + (-y)^2 = 25$, $(-x)^2 + y^2 = 25$, and $(-x)^2 + (-y)^2 = 25$ are each equivalent to $x^2 + y^2 = 25$, the graph is symmetric with respect to the $x$-axis, $y$-axis, and origin, respectively.

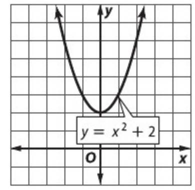




## Additional Example

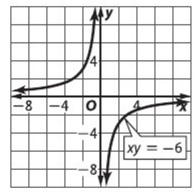
**5** Use the graph of each equation to test for symmetry with respect to the $x$-axis, the $y$-axis, and the origin. Support the answer numerically. Then confirm algebraically.

**a.**




Because $y = (-x)^2 + 2$ is equivalent to $y = x^2 + 2$, the graph is symmetric with respect to the $y$-axis.

**b.**

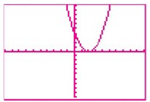



Because $(-x)(-y) = -6$ is equivalent to $xy = 6$, the graph is symmetric with respect to the origin.

## Additional Example

**6** Graph each function using a graphing calculator. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.

**a.** $f(x) = x^2 - 4x + 4$

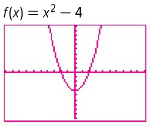


[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^2 - 4(-x) + 4$
$= x^2 + 4x + 4$

The function is neither even nor odd.

**b.** $f(x) = x^2 - 4$

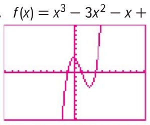


[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^2 - 4$
$= x^2 - 4$
$= f(x)$

The function is even, and the graph of the function is symmetric with respect to the *y*-axis.

**c.** $f(x) = x^3 - 3x^2 - x + 3$

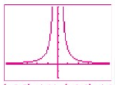


[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^3 - 3(-x)^2 - (-x) + 3$
$= x^3 - 3x^2 + x + 3$

The function is neither even nor odd.

**6A.**

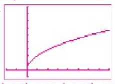


[−5, 5] scl: 1 by [−2, 8] scl: 1

$f(-x) = \frac{2}{(-x)^2}$
$= \frac{2}{x^2}$
$= f(x)$

**The function is even and the graph of the function is symmetric with respect to the *y*-axis.**

**6B.**

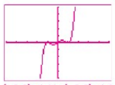


[−2, 8] scl: 1 by [−2, 18] scl: 2

$g(-x) = 4\sqrt{(-x)}$
$= 4\sqrt{-x}$

**The function is neither even nor odd.**

**6C.**



[−5, 5] scl: 1 by [−5, 5] scl: 1

$h(-x) = (-x)^5 - 2(-x)^3 + (-x)$
$= -x^5 + 2x^3 - x$
$= -h(x)$

**The function is odd, and the graph of the function is symmetric with respect to the origin.**

**StudyTip**

Even and Odd Functions It is important to always confirm symmetry algebraically. Graphs that appear to be symmetrical may not actually be.

Graphs of functions can have *y*-axis or origin symmetry. Functions with these types of symmetry have special names.

### KeyConcept Even and Odd Functions

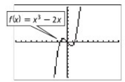
| Type of Function | Algebraic Test |
|---|---|
| Functions that are symmetric with respect to the *y*-axis are called **even functions**. | For every $x$ in the domain of $f$, $f(-x) = f(x)$. |
| Functions that are symmetric with respect to the origin are called **odd functions**. | For every $x$ in the domain of $f$, $f(-x) = -f(x)$. |

### Example 6 Identify Even and Odd Functions

GRAPHING CALCULATOR **Graph each function. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.**

**a.** $f(x) = x^3 - 2x$

It appears that the graph of the function is symmetric with respect to the origin. Test this conjecture.



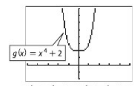

[−10, 10] scl: 1 by [−10, 10] scl: 1

$f(-x) = (-x)^3 - 2(-x)$ Substitute $-x$ for $x$.
$= -x^3 + 2x$ Simplify.
$= -(x^3 - 2x)$ Distributive Property
$= -f(x)$ Original function $f(x) = x^3 - 2x$

The function is odd because $f(-x) = -f(x)$. Therefore, the function is symmetric with respect to the origin.

**b.** $g(x) = x^4 + 2$

It appears that the graph of the function is symmetric with respect to the *y*-axis. Test this conjecture.



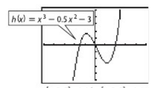

[−5, 5] scl: 1 by [−2, 8] scl: 1

$g(-x) = (-x)^4 + 2$ Substitute $-x$ for $x$.
$= x^4 + 2$ Simplify.
$= g(x)$ Original function $g(x) = x^4 + 2$

The function is even because $g(-x) = g(x)$. Therefore, the function is symmetric with respect to the *y*-axis.

**c.** $h(x) = x^3 - 0.5x^2 - 3x$

It appears that the graph of the function may be symmetric with respect to the origin. Test this conjecture algebraically.




[−5, 5] scl: 1 by [−5, 5] scl: 1

$h(-x) = (-x)^3 - 0.5(-x)^2 - 3(-x)$ Substitute $-x$ for $x$.
$= -x^3 - 0.5x^2 + 3x$ Simplify.

Because $-h(x) = -x^3 + 0.5x^2 + 3x$, the function is neither even nor odd because $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.

**GuidedPractice**

**6A.** $f(x) = \frac{2}{x^2}$ **6B.** $g(x) = 4\sqrt{x}$ **6C.** $h(x) = x^5 - 2x^3 + x$





## DifferentiatedInstruction BL

**Auditory/Musical Learners** Time a classical piece of music to determine the half-way point. Have students mark the “energy” level at 15-second intervals on graph paper. Have them mark anything that is slow or sad below the *x*-axis and anything upbeat or positive above the *x*-axis. Be sure their graphs show the half-way point being on the *y*-axis. Once students have listened and made their graphs, ask them to describe the symmetries they may see and whether the graphs are even, odd, or neither. Ask students if they like their music “even” or “odd.”

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**6** Graph each function using a graphing calculator. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.

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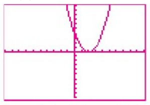

[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^2 - 4(-x) + 4$
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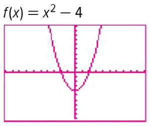
**b.** $f(x) = x^2 - 4$



[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^2 - 4$
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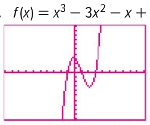
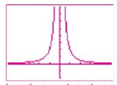


[−10,10] scl: 1 by [−10,10] scl: 1

$f(-x) = (-x)^3 - 3(-x)^2 - (-x) + 3$
$= x^3 - 3x^2 + x + 3$

The function is neither even nor odd.

**6A.**



[−5, 5] scl: 1 by [−2, 8] scl: 1

$f(-x) = \frac{2}{(-x)^2}$
$= \frac{2}{x^2}$
$= f(x)$

**The function is even and the graph of the function is symmetric with respect to the *y*-axis.**

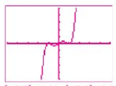
**6B.**

[−2, 8] scl: 1 by [−2, 18] scl: 2

$g(-x) = 4\sqrt{(-x)}$
$= 4\sqrt{-x}$

**The function is neither even nor odd.**

**6C.**



[−5, 5] scl: 1 by [−5, 5] scl: 1

$h(-x) = (-x)^5 - 2(-x)^3 + (-x)$
$= -x^5 + 2x^3 - x$
$= -h(x)$

**The function is odd, and the graph of the function is symmetric with respect to the origin.**

Graphs of functions can have *y*-axis or origin symmetry. Functions with these types of symmetry have special names.

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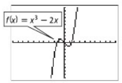
**a.** $f(x) = x^3 - 2x$

It appears that the graph of the function is symmetric with respect to the origin. Test this conjecture.




[−10, 10] scl: 1 by [−10, 10] scl: 1

$f(-x) = (-x)^3 - 2(-x)$ — Substitute $-x$ for $x$.
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$= -(x^3 - 2x)$ — Distributive Property
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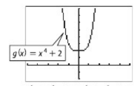
It appears that the graph of the function is symmetric with respect to the *y*-axis. Test this conjecture.



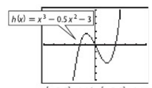

[−5, 5] scl: 1 by [−2, 8] scl: 1

$g(-x) = (-x)^4 + 2$ — Substitute $-x$ for $x$.
$= x^4 + 2$ — Simplify.
$= g(x)$ — Original function $g(x) = x^4 + 2$

The function is even because $g(-x) = g(x)$. Therefore, the function is symmetric with respect to the *y*-axis.

**c.** $h(x) = x^3 - 0.5x^2 - 3x$

It appears that the graph of the function may be symmetric with respect to the origin. Test this conjecture algebraically.




[−5, 5] scl: 1 by [−5, 5] scl: 1

$h(-x) = (-x)^3 - 0.5(-x)^2 - 3(-x)$ — Substitute $-x$ for $x$.
$= -x^3 - 0.5x^2 + 3x$ — Simplify.

Because $-h(x) = -x^3 + 0.5x^2 + 3x$, the function is neither even nor odd because $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.

**StudyTip**
**Even and Odd Functions** It is important to always confirm symmetry algebraically. Graphs that appear to be symmetrical may not actually be.

**GuidedPractice**

**6A.** $f(x) = \frac{2}{x^2}$ **6B.** $g(x) = 4\sqrt{x}$ **6C.** $h(x) = x^5 - 2x^3 + x$





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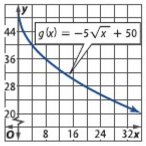
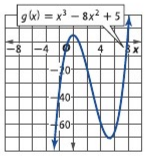
## Exercises

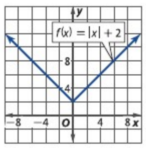
Use the graph of each function to estimate the indicated function values. Then confirm the estimate algebraically. Round to the nearest hundredth, if necessary. (Example 1)

1. 

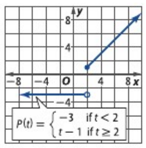

a. $g(6)$ **37.75** b. $g(12)$ **32.68** c. $g(19)$ **28.21**

2. 

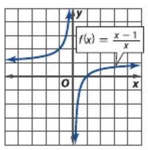

a. $g(-2)$ **−35** b. $g(1)$ **−2** c. $g(8)$ **5**

3. 

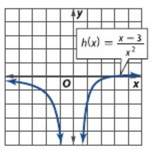

a. $f(-8)$ **10** b. $f(-3)$ **5** c. $f(0)$ **2**

4. 


a. $P(-6)$ **−3** b. $P(2)$ **1** c. $P(9)$ **8**

5. 


a. $f(-3)$ $\frac{4}{3}$ b. $f(0.5)$ **−1** c. $f(0)$ **undefined**

6. 

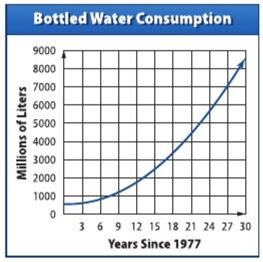

a. $h(-1)$ **−4** b. $h(1.5)$ $-\frac{2}{3}$ c. $h(2)$ $-\frac{1}{4}$

7. **RECYCLING** The quantity of paper recycled in a sample of countries in thousands of tons from 1993 to 2007 can be modeled by $p(x) = -0.0013x^4 + 0.0513x^3 - 0.662x^2 + 4.128x + 35.75$, where $x$ is the number of years since 1993. (Example 1) **a–b. See margin.**


a. Use the graph to estimate the amount of paper recycled in 1993, 1999, and 2006. Then find each value algebraically.

b. Use the graph to estimate the year in which the quantity of paper recycled reached 50,000 tons. Confirm algebraically.

8. **WATER** Bottled water consumption from 1977 to 2006 can be modeled using $f(x) = 9.35x^2 - 12.7x + 541.7$, where $x$ represents the number of years since 1977. (Example 1)

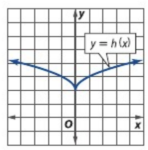


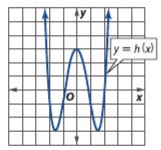

a. Use the graph to estimate the amount of bottled water consumed in 1994. **3 billion liters**

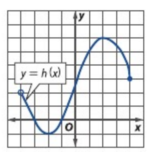
b. Find the 1994 consumption algebraically. Round to the nearest ten million liters. **3.03 billion liters**

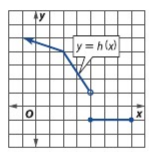
c. Use the graph to estimate when water consumption was 6 billion liters. Confirm algebraically.
**about 2002 or $x = 25$;**
**$f(24) = 9.35(24)^2 - 12.7(24) + 541.7 \approx 5623$;**
**$f(26) = 9.35(26)^2 - 12.7(26) + 541.7 \approx 6532$**

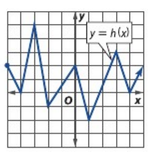
Use the graph of $h$ to find the domain and range of each function. (Example 2) **9–14. See margin.**

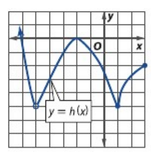
9. 


10. 


11. 


12. 


13. 


14. 






# 3 Practice

### Formative Assessment

Use Exercises 1–41 to check for understanding.

Then use the table below to customize your assignments for students.

### Exercise Alert

**Straightedge** As students estimate function values using graphs in Exercises 1–6, remind them to use a straightedge to increase accuracy.

### Additional Answers

**7a.** 36,000 tons, 46,000 tons, 54,000 tons; 35,750 tons, 46,082 tons, 53,113 tons

**7b.** about 2004 or $x = 11$; $f(10) = -0.0013(10)^4 + 0.0513(10)^3 - 0.662(10)^2 + 4.128(10) + 35.75 \approx 49$; $f(12) = -0.0013(12)^4 + 0.0513(12)^3 - 0.662(12)^2 + 4.128(12) + 35.75 \approx 52$

**9.** $D = (-\infty, \infty)$, $R = [2, \infty)$

**10.** $D = (-\infty, \infty)$, $R = [-3, \infty)$

**11.** $D = (-4, 4]$, $R = [-1, 6]$

**12.** $D = (-\infty, 7]$, $R = [-1] \cup (1, \infty)$

**13.** $D = [-5, \infty)$, $R = [-2, \infty)$

**14.** $D = (-\infty, -5) \cup (-5, 3]$, $R = [-5, \infty)$

## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–41, 69–73, 75–110 | 1–41 odd, 107–110 | 2–40 even, 69–73, 75–106 |
| OL On Level | 1–43 odd, 44–47, 49–53 odd, 54, 55, 57, 59–61, 63–67 odd, 69–73, 75–110 | 1–41, 107–110 | 42–73, 75–106 |
| BL Beyond Level | 42–110 | | |

## Additional Answers

**15a.** Sample answer: copper: D = $[-\infty, \infty]$, R = [1.75]; aluminum: D = $[-\infty, \infty]$, R = [0.6, 1.5]; zinc: D = $[-\infty, \infty]$, R = [0.5, 1.3]; steel: D = $[-\infty, \infty]$, R = [0.2, 1.75]

**15b.** Sample answer: copper ≈ 1.75 J, aluminum ≈ 1.2 J, zinc ≈ 0.5 J, steel ≈ 1.5 J

**16.** no $y$-intercept; zero: 1;

$$\sqrt{x-1} = 0$$
$$\left(\sqrt{x-1}\right)^2 = (0)^2$$
$$x - 1 = 0$$
$$x = 1$$

**17.** $y$-intercept: 0; zeros: $-1, 0, \frac{3}{2}$;

$$2x^3 - x^2 - 3x = 0$$
$$x(2x-3)(x+1) = 0$$
$$x = 0 \text{ or } 2x - 3 = 0 \text{ or } x + 1 = 0$$
$$x = \frac{3}{2} \qquad x = -1$$

**18.** $y$-intercept = 0; zero: 0;

$$\sqrt[3]{x} = 0$$
$$\left(\sqrt[3]{x}\right)^3 = (0)^3$$
$$x = 0$$

**19.** $y$-intercept = 3; no zeros;

$$\sqrt{x} + 3 = 0$$
$$\sqrt{x} \neq -3$$

**20.** $y$-intercept = 9; zero: 3;

$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$
$$x - 3 = 0$$
$$x = 3$$

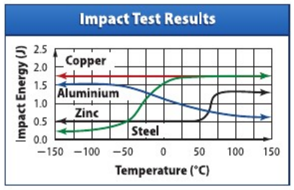
**21.** $y$-intercept = $-2$; zeros: $-\frac{1}{2}, \frac{2}{3}$;

$$6x^2 - x - 2 = 0$$
$$(2x+1)(3x-2) = 0$$
$$2x + 1 = 0 \text{ or } 3x - 2 = 0$$
$$x = -\frac{1}{2} \qquad x = \frac{2}{3}$$

**22.** $y$-intercept = 8; zero: $-2$;

$$x^3 + 6x^2 + 12x + 8 = 0$$
$$(x+2)^3 = 0$$
$$x + 2 = 0$$
$$x = -2$$

**15.** **ENGINEERING** Tests on the physical behavior of four metal specimens are performed at various temperatures in degrees Celsius. The impact energy, or energy absorbed by the sample during the test, is measured in joules. The test results are shown. (Example 2) **a–b. See margin.**




a. State the domain and range of each function.

b. Use the graph to estimate the impact energy of each metal at 0°C.

**16–23. See margin.**

**Use the graph of each function to find its $y$-intercept and zero(s). Then find these values algebraically.** (Examples 3 and 4)

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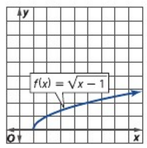


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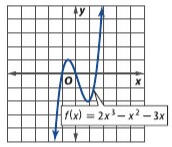


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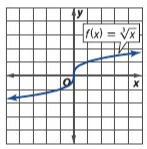


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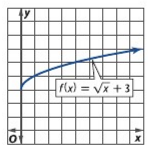


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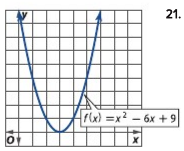


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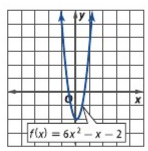


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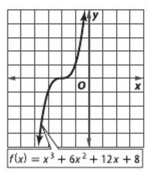


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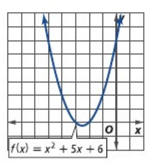


**Use the graph of each equation to test for symmetry with respect to the $x$-axis, $y$-axis, and the origin. Support the answer numerically. Then confirm algebraically.** (Example 5)

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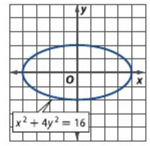


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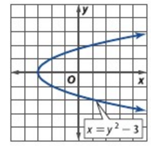


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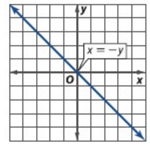


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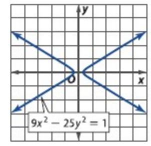


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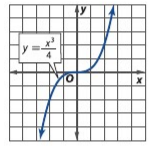


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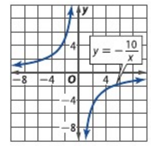


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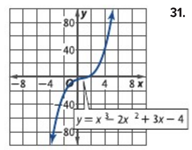


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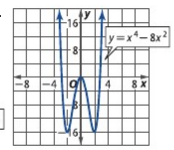


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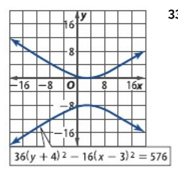


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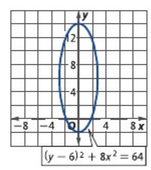



**24–33. See Chapter 11 Answer Appendix.**

**GRAPHING CALCULATOR** **Graph each function. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.** (Example 6)

**34.** $f(x) = x^2 + 6x + 10$

**35.** $f(x) = -2x^3 + 5x - 4$

**36.** $g(x) = \sqrt{x+6}$

**37.** $h(x) = \sqrt{x^2 - 9}$

**38.** $h(x) = |8 - 2x|$

**39.** $f(x) = |x^3|$

**40.** $f(x) = \frac{x+4}{x-2}$

**41.** $g(x) = \frac{x^2}{x+1}$

**34–41. See Chapter 11 Answer Appendix.**





**23.** $y$-intercept = 6; zeros: $-2, -3$;

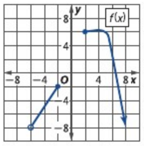
$$x^2 + 5x + 6 = 0$$
$$(x+2)(x+3) = 0$$
$$x + 2 = 0 \text{ or } x + 3 = 0$$
$$x = -2 \qquad x = -3$$

### WatchOut!

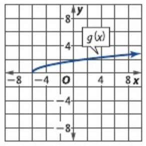
**Problem Context** In Exercise 15a, some students may notice that the domain for the graphs should not be $[-\infty, \infty]$. The coldest possible temperature is absolute zero, at approximately -273.15 degrees Celsius. Therefore, the true domain of each graph is [-273.15, ∞]

Use the graph of each function to estimate the indicated function values. **42a. −2 42b. undefined 42c. undefined**

42.



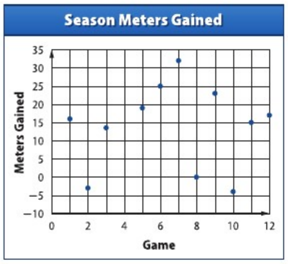

43.




a. $f(-2)$ b. $f(-6)$ c. $f(0)$ a. $g(-8)$ b. $g(-6)$ c. $g(-2)$

**undefined 0 1.5**

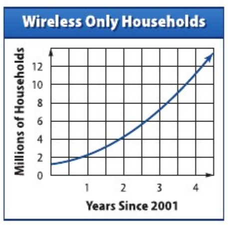
44. **RUGBY** A fullback's meters gained for each game in a season are shown. **a. See margin.**




a. State the domain and range of the relation.

b. In what game did the player not gain any meters? **Game 8**

45. **PHONES** The number of households $h$ in millions with only wireless phone service from 2001 to 2005 can be modeled by $h(x) = 0.5x^2 + 0.5x + 1.2$, where $x$ represents the number of years after 2001. **a–d. See margin.**




a. State the relevant domain and approximate the range.

b. Use the graph to estimate the number of households with only wireless phone service in 2003. Then find it algebraically.

c. Use the graph to approximate the $y$-intercept of the function. Then find it algebraically. What does the $y$-intercept represent?

d. Does this function have any zeros? If so, estimate them and explain their meaning. If not, explain why.

46. **FUNCTIONS** Consider $f(x) = x^n$. **a–d. See Chapter 11 Answer Appendix.**

a. Use a graphing calculator to graph $f(x)$ for values of $n$ in the range $1 \le n \le 6$, where $n \in \mathbb{N}$.

b. Describe the domain and range of each function.

c. Describe the symmetry of each function.

d. Predict the domain, range, and symmetry for $f(x) = x^{35}$. Explain your reasoning.

47. **PHARMACOLOGY** Suppose the number of milligrams of a pain reliever in the bloodstream $x$ hours after taking a dose is modeled by $f(x) = 0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x$. **47a–b. See margin.**

a. Use a graphing calculator to graph the function.

b. State the relevant domain. Explain your reasoning.

c. What was the approximate maximum amount of pain reliever, in milligrams, in the bloodstream at any given time? **about 346 mg**

**GRAPHING CALCULATOR** **Graph each function and locate the zeros for each function. Confirm your answers algebraically.**

48. $f(x) = \frac{4x - 1}{x}$

49. $f(x) = \frac{x^2 + 9}{x + 3}$

50. $h(x) = \sqrt{x^2 + 4x + 3}$

51. $h(x) = 2\sqrt{x + 12} - 8$

52. $g(x) = -12 + \frac{4}{x}$

53. $g(x) = \frac{6}{x} + 3$

**48–53. See Chapter 11 Answer Appendix.**

54. **TELEVISION** In a city, the percent of households $h$ with basic cable for the years 1986 through 2012 can be modeled using $h(x) = -0.115x^2 + 4.43x + 25.6$, $0 \le x \le 26$, where $x$ represents the number of years after 1986. **See margin.**

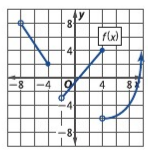
a. Use a graphing calculator to graph the function.

b. What percent of households had basic cable in 2005? Round to the nearest percent. **68%**

c. For what years was the percent of subscribers greater than 65%? **2000 through 2010**

**Use the graph of $f$ to find the domain and range of each function.** **55–58. See margin.**

55.

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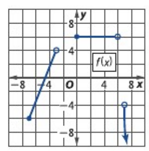
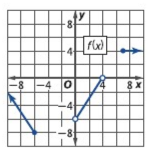
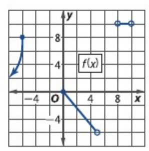



57.




58.








**WatchOut!**

**Common Error** In Exercise 44, some students may describe the domain of the relation as [1, 12]. Remind students that a relation with a domain that is a set of individual points is *discrete*. The domain of a discrete function cannot be described by an interval that includes an infinite number of real values.

## Additional Answers

**44a.** D = {1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12}

R = {16, −3, 13, 19, 25, 32, 0, 23, −4, 15 ,17}

**45a.** $D = \{x \mid 0 \le x \le 4, x \in \mathbb{R}\}$, $R = \{y \mid 1.2 \text{ million} \le y \le 11.2 \text{ million}, y \in \mathbb{R}\}$

**45b.** Sample answers: 4.1 million, 4.2 million

**45c.** Sample answers: 1.1, 1.2; The $y$-intercept represents the number of millions of households with only wireless phone service in 2001.

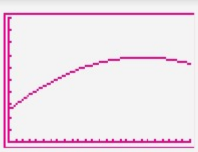
**45d.** No; sample answer: There were more than zero households with only wireless phone service for all of the years in the domain.

**47a.**

[−5, 15] scl: 2 by [−100, 400] scl: 50

**47b.** [0, 6]; The relevant domain represents the interval of time beginning when the dose was first taken and ending when the pain reliever left the bloodstream. Because time cannot be negative, $x \ge 0$. The amount of pain reliever in the bloodstream is zero when the dose is first taken at $x = 0$, and is zero again at $x = 6$. Therefore, $x \le 6$. Combining these restrictions, the relevant domain is $D = \{x \mid 0 \le x \le 6, x \in \mathbb{R}\}$ or [0, 6].

**54a.**



[0, 26] scl: 2 by [0, 100] scl: 10

**55.** $D = (-8, -4] \cup (-2, \infty)$, $R = (-6, \infty)$

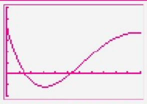
**56.** $D = [-7, -3) \cup [0, 6) \cup (7, \infty)$, $R = (-\infty, 4) \cup [6]$

**57.** $D = (-\infty, -6] \cup (0, 4) \cup [7, \infty)$, $R = [-8, \infty)$

**58.** $D = (-\infty, -6] \cup [0, 5) \cup (8, 10)$, $R = (-\infty, 8] \cup [10]$

## Additional Answers

**60a.**



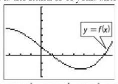
[0, 11] scl: 1 by [−0.5, 1.5] scl: 0.25

**60b.** D = [0, 11], R = [−0.3, 1.04]

**60c.** 1.04; Sample answer: The $y$-intercept represents the initial stock fluctuation percentage.

**60d.** 1.5, 5.2; The zeros represent the months when the stock price returned to its original value.

**59.** **POPULATION** At the beginning of 1900 a city had a population of 140,000. The percent change from 1930 to 2010 is modeled by $f(x) = 0.0001x^3 - 0.001x^2 - 0.825x + 12.58$, where $x$ is the number of years since 1930.




[−50, 100] scl: 15 by [−30, 70] scl: 10

a. State the relevant domain and estimate the range for this domain.

b. Use the graph to approximate the $y$-intercept. Then find the $y$-intercept algebraically. What does the $y$-intercept represent?

c. Find and interpret the zeros of the function.

d. Use the model to determine what the percent population change will be in 2080. Does this value seem realistic? Explain your reasoning.

**a–d. See Chapter 11 Answer Appendix.**

**60.** **STOCK MARKET** The percent $p$ a stock price has fluctuated in one year can be modeled by $p(x) = 0.0005x^4 - 0.0193x^3 + 0.243x^2 - 1.014x + 1.04$, where $x$ is the number of months since January. **a–d. See margin.**

a. Use a graphing calculator to graph the function.

b. State the relevant domain and estimate the range.

c. Use the graph to approximate the $y$-intercept. Then find the $y$-intercept algebraically. What does the $y$-intercept represent?

d. Find and interpret any zeros of the function.

**61.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the range values of $f(x) = \frac{1}{x-2}$ as $x$ approaches 2. **b–d. See Chapter 11 Answer Appendix.**

a. **TABULAR** Copy and complete the table below. Add an additional value to the left and right of 2.

| $x$ | 1.99 | 1.999 | 2 | 2.001 | 2.01 |
|---|---|---|---|---|---|
| $f(x)$ | −100 | −1000 | undefined | 1000 | 100 |

b. **ANALYTICAL** Use the table from part **a** to describe the behavior of the function as $x$ approaches 2.

c. **GRAPHICAL** Graph the function. Does the graph support your conjecture from part **b**? Explain.

d. **VERBAL** Make a conjecture as to why the graph of the function approaches the value(s) found in part **c**, and explain any inconsistencies present in the graph.

**62–69. See Chapter 11 Answer Appendix.**

**GRAPHING CALCULATOR** **Graph each function. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.**

**62.** $f(x) = x^2 - x - 6$

**63.** $g(n) = n^2 - 37$

**64.** $h(x) = x^6 + 4$

**65.** $f(g) = g^9$

**66.** $g(y) = y^4 + 8y^2 + 81$

**67.** $h(y) = y^5 - 17y^3 + 16y$

**68.** $h(b) = b^4 - 2b^3 - 13b^2 + 14b + 24$

## H.O.T. Problems Use Higher-Order Thinking Skills

**OPEN ENDED** **Sketch a graph that matches each description.**

**69.** passes through (−3, 8), (−4, 4), (−5, 2), and (−8, 1) and is symmetric with respect to the $y$-axis

**70.** passes through (0, 0), (2, 6), (3, 12), and (4, 24) and is symmetric with respect to the $x$-axis

**71.** passes through (−3, −18), (−2, −9), and (−1, −3) and is symmetric with respect to the origin

**72.** passes through (4, −16), (6, −12), and (8, −8) and represents an even function

**69–72. See Chapter 11 Answer Appendix.**

**73.** **WRITING IN MATH** Explain why a function can have 0, 1, or more $x$-intercepts but only one $y$-intercept.

**See Chapter 11 Answer Appendix.**

**74.** **CHALLENGE** Use a graphing calculator to graph $f(x) = \frac{2x^2 + 3x - 2}{x^3 - 4x^2 - 12x}$, and predict its domain. Then confirm the domain algebraically. Explain your reasoning.

**See Chapter 11 Answer Appendix.**

**REASONING** **Determine whether each statement is *true* or *false*. Explain your reasoning.**

**75.** The range of $f(x) = nx^2$, where $n$ is any integer, is $\{y \mid y \geq 0, y \in \mathbb{R}\}$.

**76.** The range of $f(x) = \sqrt{nx}$, where $n$ is any integer, is $\{y \mid y \geq 0, y \in \mathbb{R}\}$.

**77.** All odd functions are also symmetric with respect to the line $y = -x$.

**78.** An even function rotated $180n°$ about the origin, where $n$ is any integer, remains an even function.

**75–78. See Chapter 11 Answer Appendix.**

**REASONING** **If $a(x)$ is an odd function, determine whether $b(x)$ is *odd*, *even*, *neither*, or *cannot be determined*. Explain your reasoning.** **79–83. See Chapter 11 Answer Appendix.**

**79.** $b(x) = a(-x)$

**80.** $b(x) = -a(x)$

**81.** $b(x) = [a(x)]^2$

**82.** $b(x) = a(|x|)$

**83.** $b(x) = [a(x)]^3$

**84–87. See Chapter 11 Answer Appendix.**

**REASONING** **State whether a graph with each type of symmetry *always*, *sometimes*, or *never* represents a function. Explain your reasoning.**

**84.** symmetric with respect to the line $x = 4$

**85.** symmetric with respect to the line $y = 2$

**86.** symmetric with respect to the line $y = x$

**87.** symmetric with respect to both the $x$- and $y$-axes

**See Chapter 11 Answer Appendix.**

**88.** **WRITING IN MATH** Can a function be both even and odd? Explain your reasoning.





**718** | **Lesson 11-2** | Analyzing Graphs of Functions and Relations

## Spiral Review

Find each function value. (Lesson 11-1)

**89.** $g(x) = x^2 - 10x + 3$
- **a.** $g(2)$ $-13$
- **b.** $g(-4x)$ $16x^2 + 40x + 3$
- **c.** $g(1 + 3n)$ $9n^2 - 24n - 6$

**90.** $h(x) = 2x^2 + 4x - 7$
- **a.** $h(-9)$ $119$
- **b.** $h(3x)$ $18x^2 + 12x - 7$
- **c.** $h(2 + m)$ $2m^2 + 12m + 9$

**91.** $p(x) = \frac{2x^3 + 2}{x^2 - 2}$
- **a.** $p(3)$ $8$
- **b.** $p(x^2)$ $\frac{2x^6 + 2}{x^4 - 2}$
- **c.** $p(x + 1)$ $\frac{2x^3 + 6x^2 + 6x + 4}{x^2 + 2x - 1}$

**92.** **GRADES** The midterm grades for a Chemistry class of 25 students are shown. Find the measures of spread for the data set.
range = 36, variance ≈ 92.57, standard deviation ≈ 9.62

| Midterm Grades | | | | |
|---|---|---|---|---|
| 89 | 76 | 91 | 72 | 81 |
| 81 | 65 | 74 | 80 | 74 |
| 73 | 92 | 76 | 83 | 96 |
| 66 | 61 | 80 | 74 | 70 |
| 97 | 78 | 73 | 62 | 72 |

**93.** **FLASHCARDS** From a set of 52 flashcards, divided equally between four different colors, red, yellow, green and blue with each color numbered 1 to 13 find how many 5-flashcard selections are possible that fit each description.
- **a.** 3 reds and 2 yellows 22,308
- **b.** 1 1, 2 11s, and 2 13s 144
- **c.** all 5 with numbers 11, 12 or 13 792

Find the following for $A = \begin{bmatrix} -6 & 3 \\ -5 & 11 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -7 \\ 2 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

**94.** $4A - 2B$ $\begin{bmatrix} -30 & 26 \\ -24 & 50 \end{bmatrix}$

**95.** $3C + 2A$ no solution

**96.** $-2(B - 3A)$ $\begin{bmatrix} -42 & 32 \\ -34 & 72 \end{bmatrix}$

Evaluate each expression.

**97.** $27^{\frac{1}{3}}$ $3$

**98.** $64^{\frac{5}{6}}$ $32$

**99.** $49^{-\frac{1}{2}}$ $\frac{1}{7}$

**100.** $16^{-\frac{3}{4}}$ $\frac{1}{8}$

**101.** $25^{\frac{3}{2}}$ $125$

**102.** $36^{-\frac{3}{2}}$ $\frac{1}{216}$

**103.** **GENETICS** Suppose $R$ and $W$ represent two genes that a plant can inherit from its parents. The terms of the expansion of $(R + W)^2$ represent the possible pairings of the genes in the offspring. Write $(R + W)^2$ as a polynomial. $R^2 + 2RW + W^2$

Simplify.

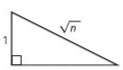
**104.** $(2 + i)(4 + 3i)$ $5 + 10i$

**105.** $(1 + 4i)^2$ $-15 + 8i$

**106.** $(2 - i)(3 + 2i)(1 - 4i)$ $12 - 31i$

## Skills Review for Standardized Tests

**107.** **SAT/ACT** In the figure, if $n$ is a real number greater than 1, what is the value of $x$ in terms of $n$? B




- A $\sqrt{n^2 - 1}$
- B $\sqrt{n - 1}$
- C $\sqrt{n + 1}$
- D $n - 1$
- E $n + 1$

**108.** **REVIEW** Which inequality describes the range of $f(x) = x^2 + 1$ over the domain $-2 < x < 3$? J
- F $5 \le y < 9$
- G $2 < y < 10$
- H $1 < y < 9$
- J $1 \le y < 10$

**109.** Which of the following is an even function? B
- A $f(x) = 2x^4 + 6x^3 - 5x^2 - 8$
- B $g(x) = 3x^6 + x^4 - 5x^2 + 15$
- C $m(x) = x^4 + 3x^3 + x^2 + 35x$
- D $h(x) = 4x^6 + 2x^4 + 6x - 4$

**110.** Which of the following is the domain of $g(x) = \frac{1 + x}{x^2 - 16x}$? F
- F $(-\infty, 0) \cup (0, 16) \cup (16, \infty)$
- G $(-\infty, 0] \cup [16, \infty)$
- H $(-\infty, -1) \cup (-1, \infty)$
- J $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$





## 4 Assess

**Name the Math** List the steps required to algebraically check whether a function is *even*, *odd*, or *neither*.

1. If $f(-x) = f(x)$, then the function is even.
2. If $f(-x) = -f(x)$, then the function is odd.
3. If $f(-x) \neq f(x)$ or $-f(x)$, then the function is neither even nor odd.

## Differentiated Instruction BL

**Extension** Suppose $f(x)$ is an even function and $g(x)$ is an odd function. If $h(x) = f(x) \cdot g(x)$, is $h(x)$ an even function, an odd function, or neither? Justify your answer. Odd; sample answer:

$$\begin{aligned} h(-x) &= f(-x) \cdot g(-x) \\ &= f(x) \cdot -(g(x)) \\ &= f(x) \cdot g(x) \\ &= -h(x) \end{aligned}$$

# 1 Focus

## VerticalAlignment

**Before Lesson 11-3** Find domain and range using the graph of a function.

**Lesson 11-3** Use limits to determine the continuity of a function, and apply the Intermediate Value Theorem to continuous functions.
Use limits to describe end behavior of a function.

**After Lesson 11-3** Find extrema of a function.

# 2 Teach

## Scaffolding Questions

Have students read the **Why?** section of the lesson and examine the graph.

**Ask:**

- What was the minimum wage in 1984? approximately AED 3.50
- What was the minimum wage at the beginning of 1996? almost AED 5

# LESSON 11-3 Continuity, End Behavior, and Limits

**Then**

- You found domain and range using the graph of a function. (Lesson 11-2)

**Now**

1. Use limits to determine the continuity of a function, and apply the Intermediate Value Theorem to continuous functions.
2. Use limits to describe end behavior of functions.

**Why?**

- Since the early 1980s, the current minimum wage in the U.S. has jumped up several times. The graph of the minimum wage as a function of time shows these jumps as breaks in the graph, such as those at $x = 1990$, $x = 1996$, and $x = 2008$.

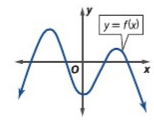
**Federal Minimum Wage**


## NewVocabulary

continuous function
limit
discontinuous function
infinite discontinuity
jump discontinuity
removable discontinuity
nonremovable discontinuity
end behavior

## 1 Continuity

The graph of a **continuous function** has no breaks, holes, or gaps. You can trace the graph of a continuous function without lifting your pencil.



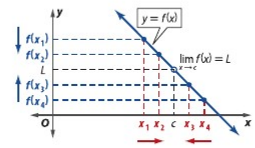
$f(x)$ is continuous for all $x$.

One condition for a function $f(x)$ to be continuous at $x = c$ is that the function must approach a unique function value as $x$-values approach $c$ from the left and right sides. The concept of approaching a value without necessarily ever reaching it is called a **limit**.

### KeyConcept Limits

**Words** If the value of $f(x)$ approaches a unique value $L$ as $x$ approaches $c$ from each side, then the limit of $f(x)$ as $x$ approaches $c$ is $L$.

**Symbols** $\lim_{x \to c} f(x) = L$, which is read *The limit of f(x) as x approaches c is L.*

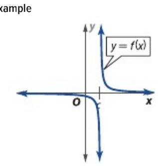



To understand what it means for a function to be continuous from an algebraic perspective, it helps to examine the graphs of **discontinuous functions**, or functions that are not continuous. Functions can have many different types of discontinuity.

### KeyConcept Types of Discontinuity

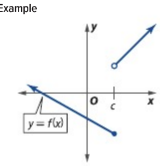
A function has an **infinite discontinuity** at $x = c$ if the function value increases or decreases indefinitely as $x$ approaches $c$ from the left and right.

Example



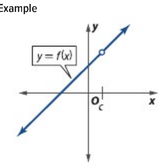

A function has a **jump discontinuity** at $x = c$ if the limits of the function as $x$ approaches $c$ from the left and right exist but have two distinct values.

Example




A function has a **removable discontinuity** if the function is continuous everywhere except for a hole at $x = c$.

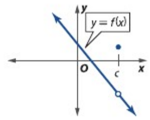
Example



**StudyTip**

**Limits** Whether $f(x)$ exists at $x = c$ has no bearing on the existence of the *limit* of $f(x)$ as $x$ approaches $c$.

Notice that for graphs of functions with a removable discontinuity, the limit of $f(x)$ at point $c$ exists, but either the value of the function at $c$ is undefined, or, as with the graph shown, the value of $f(c)$ is not the same as the value of the limit at point $c$.




Infinite and jump discontinuities are classified as **nonremovable discontinuities**. A nonremovable discontinuity cannot be eliminated by redefining the function at that point, since the function approaches different values from the left and right sides at that point or does not approach a single value at all. Instead it is increasing or decreasing indefinitely.

These observations lead to the following test for the continuity of a function.

**ConceptSummary** Continuity Test

A function $f(x)$ is continuous at $x = c$ if it satisfies the following conditions.

- $f(x)$ is defined at $c$. That is, $f(c)$ exists.
- $f(x)$ approaches the same value from either side of $c$. That is, $\lim_{x \to c} f(x)$ exists.
- The value that $f(x)$ approaches from each side of $c$ is $f(c)$. That is, $\lim_{x \to c} f(x) = f(c)$.

**Example 1** Identify a Point of Continuity

**Determine whether $f(x) = 2x^2 - 3x - 1$ is continuous at $x = 2$. Justify using the continuity test.**

Check the three conditions in the continuity test.

**1. Does $f(2)$ exist?**

Because $f(2) = 1$, the function is defined at $x = 2$.

**2. Does $\lim_{x \to 2} f(x)$ exist?**

Construct a table that shows values of $f(x)$ for $x$-values approaching 2 from the left and from the right.

| | $x$ approaches 2 → | | | | ← $x$ approaches 2 | | |
|---|---|---|---|---|---|---|---|
| $x$ | 1.9 | 1.99 | 1.999 | 2.0 | 2.001 | 2.01 | 2.1 |
| $f(x)$ | 0.52 | 0.95 | 0.995 | | 1.005 | 1.05 | 1.52 |

The pattern of outputs suggests that as the value of $x$ gets closer to 2 from the left and from the right, $f(x)$ gets closer to 1. So, we estimate that $\lim_{x \to 2} f(x) = 1$.

**3. Does $\lim_{x \to 2} f(x) = f(2)$?**

Because $\lim_{x \to 2} (2x^2 - 3x - 1)$ is estimated to be 1 and $f(2) = 1$, we conclude that $f(x)$ is continuous at $x = 2$. The graph of $f(x)$ shown in Figure 13.3.1 supports this conclusion.

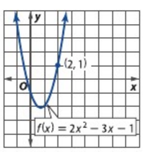



Figure 13.3.1

**GuidedPractice**

**Determine whether each function is continuous at $x = 0$. Justify using the continuity test.**

**1A.** $f(x) = x^3$

**1B.** $f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

**1A. Continuous at $x = 0$; $f(0) = 0$, $\lim_{x \to 0} f(x) = 0$, and $\lim_{x \to 0} f(x) = f(0)$.**

**1B. Discontinuous at $x = 0$; $f(x)$ approaches $-\infty$ as $x$ approaches 0 from the left and 0 as $x$ approaches 0 from the right.**





- What do the closed circles on the graph indicate? What do the open circles on the graph indicate? Closed circles show that the function has an endpoint included at that point. Open circles show that the function has an endpoint that is not included at that point.

## 1 Continuity

**Examples 1 and 2** show how to identify points of continuity and discontinuity for functions. **Example 3** shows how to approximate the zeros of a function on a given interval.

### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

**Additional Example**

**1** Determine whether $f(x) = \frac{1}{2x + 1}$ is continuous at $x = \frac{1}{2}$. Justify using the continuity test. Continuous at $x = \frac{1}{2}$; $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$, and $\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$.

721

**Additional Example**

**2** Determine whether each function is continuous at the given $x$-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.

**a.** $f(x) = \frac{1}{x-1}$; at $x = 1$

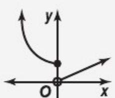
Discontinuous; because $f(1)$ is undefined and $f(x)$ decreases indefinitely as $x$ approaches 1 from the left and increases indefinitely as $x$ approaches 1 from the right, $f(x)$ has an infinite discontinuity at $x = 1$.

**b.** $f(x) = \frac{x-2}{x^2-4}$; at $x = 2$

Discontinuous; because $f(2)$ is undefined and $f(x)$ approaches 0.25 as $x$ approaches 2 from the left and the right, $f(x)$ has a removable discontinuity at $x = 2$.

## Focus on Mathematical Content

**Continuity** Other types of jump discontinuity include



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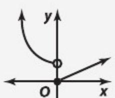
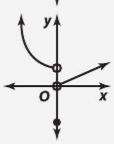
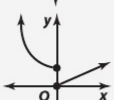


, and


, but not



since it is not a function.

Similarly, when a discontinuity is removable, the function may or may not be defined at that point.



If just one of the conditions for continuity is not satisfied, the function is discontinuous at $x = c$. Examining a function can help you identify the type of discontinuity at that point.

### Example 2 Identify a Point of Discontinuity

**Determine whether each function is continuous at the given $x$-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.**

a. $f(x) = \begin{cases} 3x - 2 & \text{if } x > -3 \\ 2 - x & \text{if } x \le -3 \end{cases}$; at $x = -3$

**1.** Because $f(-3) = 5$, $f(-3)$ exists.

**2.** Investigate function values close to $f(-3)$.

*x* approaches −3 → ← *x* approaches −3

| $x$ | −3.1 | −3.01 | −3.001 | −3.0 | −2.999 | −2.99 | −2.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 5.1 | 5.01 | 5.001 | | −10.997 | −10.97 | 10.7 |

The pattern of outputs suggests that $f(x)$ approaches 5 as $x$ approaches −3 from the left and −11 as $f(x)$ approaches −3 from the right. Because these values are not the same, $\lim_{x \to -3} f(x)$ does not exist. Therefore, $f(x)$ is discontinuous at $x = -3$. Because $f(x)$ approaches two different values when $x = -3$, $f(x)$ has a jump discontinuity at $x = -3$. The graph of $f(x)$ in Figure 13.3.2 supports this conclusion.

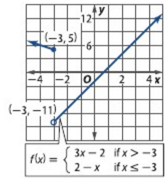



Figure 13.3.2

b. $f(x) = \frac{x+3}{x^2-9}$; at $x = -3$ and $x = 3$

**1.** Because $f(-3) = \frac{0}{0}$ and $f(3) = \frac{6}{0}$, both of which are undefined, $f(-3)$ and $f(3)$ do not exist. Therefore, $f(x)$ is discontinuous at both $x = -3$ and at $x = 3$.

**2.** Investigate function values close to $f(-3)$.

*x* approaches −3 → ← *x* approaches −3

| $x$ | −3.1 | −3.01 | −3.001 | −3.0 | −2.999 | −2.99 | −2.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −0.164 | −0.166 | −0.167 | | −0.167 | −0.167 | −0.169 |

The pattern of outputs suggests that $f(x)$ approaches a limit close to −0.167 as $x$ approaches −3 from each side, so $\lim_{x \to -3} f(x) \approx -0.167$ or $-\frac{1}{6}$.

Investigate function values close to $f(3)$.

*x* approaches 3 → ← *x* approaches 3

| $x$ | 2.9 | 2.99 | 2.999 | 3.0 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −10 | −100 | −1000 | | 1000 | 100 | 10 |

The pattern of outputs suggests that for values of $x$ approaching 3 from the left, $f(x)$ becomes increasingly more negative. For values of $x$ approaching 3 from the right, $f(x)$ becomes increasingly more positive. Therefore, $\lim_{x \to 3} f(x)$ does not exist.

**3.** Because $\lim_{x \to -3} f(x)$ exists, but $f(-3)$ is undefined, $f(x)$ has a removable discontinuity at $x = -3$. Because $f(x)$ decreases without bound as $x$ approaches 3 from the left and increases without bound as $x$ approaches 3 from the right, $f(x)$ has an infinite discontinuity at $x = 3$. The graph of $f(x)$ in Figure 13.3.3 supports these conclusions.

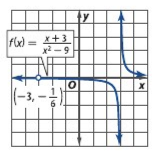



Figure 13.3.3

**Guided Practice**

**2A.** $f(x) = \frac{1}{x^2}$; at $x = 0$

**2B.** $f(x) = \begin{cases} 5x + 4 & \text{if } x > 2 \\ 2 - x & \text{if } x \le 2 \end{cases}$; at $x = 2$

**2A. Discontinuous; because $f(0)$ is undefined and $f(x)$ increases indefinitely as $x$ approaches 0 from the left and the right, $f(x)$ has an infinite discontinuity at $x = 0$.**

**2B. Discontinuous; because $f(x)$ approaches 0 as $x$ approaches 2 from the left and 14 as $x$ approaches 2 from the right, $f(x)$ has a jump discontinuity at $x = 2$.**

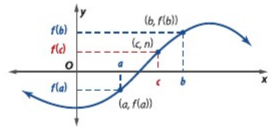


**722** | **Lesson 11-3** | Continuity, End Behavior, and Limits

If a function is continuous, you can approximate the location of its zeros by using the Intermediate Value Theorem and its corollary The Location Principle.

### KeyConcept Intermediate Value Theorem

If $f(x)$ is a continuous function and $a < b$ and there is a value $n$ such that $n$ is between $f(a)$ and $f(b)$, then there is a number $c$, such that $a < c < b$ and $f(c) = n$.




**Corollary: The Location Principle** If $f(x)$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$, such that $a < c < b$ and $f(c) = 0$. That is, there is a zero between $a$ and $b$.

### Example 3 Approximate Zeros

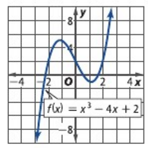
**Determine between which consecutive integers the real zeros of each function are located on the given interval.**

**a.** $f(x) = x^3 - 4x + 2$; $[-4, 4]$

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | −46 | **−13** | **2** | 5 | **2** | **−1** | **2** | 17 | 50 |

Because $f(-3)$ is negative and $f(-2)$ is positive, by the Location Principle, $f(x)$ has a zero between $-3$ and $-2$. The value of $f(x)$ also changes sign for $0 \le x \le 1$ and $1 \le x \le 2$. This indicates the existence of real zeros in each of these intervals.

The graph of $f(x)$ shown at the right supports the conclusion that there are real zeros between $-3$ and $-2$, 0 and 1, and 1 and 2.

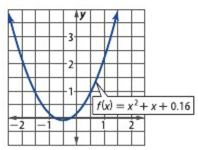



**b.** $f(x) = x^2 + x + 0.16$; $[-3, 3]$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 6.16 | 2.16 | 0.16 | 0.16 | 2.16 | 6.16 | 12.16 |

The values of $f(x)$ do not change sign for the $x$-values used. However, as the $x$-values approach $-1$ from the left, $f(x)$ decreases, then begins increasing at $x = 0$. So, there may be real zeros between consecutive integers $-1$ and 0. Graph the function to verify.

The graph of $f(x)$ crosses the $x$-axis twice on the interval $[-1, 0]$, so there are real zeros between $-1$ and 0.




**StudyTip**

**Approximating Zeros with No Sign Changes** While a sign change on an interval *does* indicate the location of a real zero, the absence of a sign change *does not* indicate that there are no real zeros on that interval. The best method of checking this is to graph the function.

**GuidedPractice**

**3A.** $f(x) = \dfrac{x^2 - 6}{x + 4}$; $[-3, 4]$ **−3 and −2, 2 and 3** **3B.** $f(x) = 8x^3 - 2x^2 - 5x - 1$; $[-5, 0]$ **−1 and 0**





### Additional Example

**3** Determine between which consecutive integers the real zeros of each function are located on the given interval.

**a.** $f(x) = x^2 - x - \frac{3}{4}$; $[-2, 2]$
−1 and 0, 1 and 2

**b.** $f(x) = x^3 + 2x + 5$; $[-2, 2]$
−2 and −1

### Tips for New Teachers

**Sign Changes** Explain to students that there is no sign change for $f(x) = (x - 1)^2$ but there is a real zero of multiplicity 2 at $x = 1$.

### WatchOut!

**Windows** In the standard viewing window, there appears to be only one intersection for Example 4. Use ZDecimal or Zoom In and adjust the window in order to get a closer look at the zeros.

### Teach with Tech

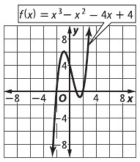
**Graphing Calculator** Advise students that the zeros of a function can be found by selecting zero from the CALC menu. Prompts will ask for the Left Bound, which is any $x$-value less than the zero, and the Right Bound, which is any $x$-value greater than the zero. Explain that the calculator uses the Location Principle to determine if there is a sign change, and thus a zero, between the corresponding function values.

## 2 End Behavior

**Example 4** shows how to describe end behavior when a function approaches infinity. **Examples 5 and 6** show how to describe the end behavior when a function approaches a specific value.

### Additional Example

**4** Use the graph of $f(x) = x^3 - x^2 - 4x + 4$ to describe its end behavior. Support the conjecture numerically.




From the graph, it appears that as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

| $x$ | $f(x)$ |
|---|---|
| −10,000 | $-1 \times 10^{12}$ |
| −1000 | $-1 \times 10^{9}$ |
| 0 | 4 |
| 1000 | $1 \times 10^{9}$ |
| 10,000 | $1 \times 10^{12}$ |

### Guided Practice

**4A.** From the graph, it appears that as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to -\infty$.

| $x$ | $g(x)$ |
|---|---|
| −10,000 | $-1 \times 10^{12}$ |
| −1000 | $-1 \times 10^{9}$ |
| 0 | 2 |
| 1000 | $1 \times 10^{9}$ |
| 10,000 | $1 \times 10^{12}$ |

**4B.** From the graph, it appears that as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$.

| $x$ | $f(x)$ |
|---|---|
| −10,000 | $2.5 \times 10^{11}$ |
| −1000 | $2.5 \times 10^{8}$ |
| 0 | 0 |
| 1000 | $-2.5 \times 10^{8}$ |
| 10,000 | $-2.5 \times 10^{11}$ |

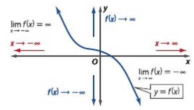
**2 End Behavior** The **end behavior** of a function describes how a function *behaves* at either *end* of the graph. That is, end behavior is what happens to the value of $f(x)$ as $x$ increases or decreases without bound—becoming greater and greater or more and more negative. To describe the end behavior of a graph, you can use the concept of a limit.

**ReadingMath**

**Limits** The expression $\lim_{x \to \infty} f(x)$ is read *the limit of f(x) as x approaches positive infinity.* The expression $\lim_{x \to -\infty} f(x)$ is read *the limit of f(x) as x approaches negative infinity.*

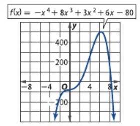
| Left-End Behavior | Right-End Behavior |
|---|---|
| $\lim_{x \to -\infty} f(x)$ | $\lim_{x \to \infty} f(x)$ |

One possibility for the end behavior of the graph of a function is for the value of $f(x)$ to increase or decrease without bound. This end behavior is described by saying that $f(x)$ approaches positive or negative infinity.




### Example 4 Graphs that Approach Infinity

**Use the graph of $f(x) = -x^4 + 8x^3 + 3x^2 + 6x - 80$ to describe its end behavior. Support the conjecture numerically.**




**Analyze Graphically**

In the graph of $f(x)$, it appears that $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$.

**Support Numerically**

Construct a table of values to investigate function values as $|x|$ increases. That is, investigate the value of $f(x)$ as the value of $x$ becomes greater and greater or more and more negative.

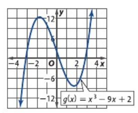
| | ← $x$ approaches $-\infty$ | | | | $x$ approaches $\infty$ → | | |
|---|---|---|---|---|---|---|---|
| $x$ | −10,000 | −1000 | −100 | 0 | 100 | 1000 | 10,000 |
| $f(x)$ | $-1 \cdot 10^{16}$ | $-1 \cdot 10^{12}$ | $-1 \cdot 10^{8}$ | −80 | $-1 \cdot 10^{8}$ | $-1 \cdot 10^{12}$ | $-1 \cdot 10^{16}$ |

The pattern of outputs suggests that as $x$ approaches $-\infty$, $f(x)$ approaches $-\infty$ and as $x$ approaches $\infty$, $f(x)$ approaches $-\infty$. This supports the conjecture.

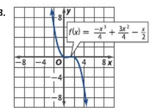
**Guided Practice**

**Use the graph of each function to describe its end behavior. Support the conjecture numerically.** **4A–B. See margin.**

**4A.**




**4B.**




Instead of $f(x)$ being unbounded, approaching $\infty$ or $-\infty$ as $|x|$ increases, some functions approach, but never reach, a fixed value.



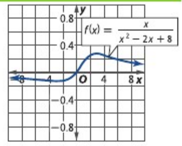


### DifferentiatedInstruction AL OL

**Logical Learners** Ask students to develop or recall general rules for graphing functions. Have students test their rules by first plotting graphs without a graphing tool and then using a graphing tool to check their assumptions. Ask students to consider what determines vertical and horizontal asymptotes.

## Example 5 Graphs that Approach a Specific Value

**Use the graph of $f(x) = \frac{x}{x^2 - 2x + 8}$ to describe its end behavior. Support the conjecture numerically.**




**Analyze Graphically**

In the graph of $f(x)$, it appears that $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$.

**Support Numerically**

← x approaches −∞ | x approaches ∞ →

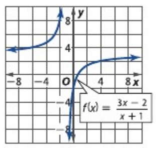
| $x$ | −10,000 | −1000 | −100 | 0 | 100 | 1000 | 10,000 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1 \cdot 10^{-4}$ | −0.001 | −0.01 | 0 | 0.01 | 0.001 | $1 \cdot 10^{-4}$ |

The pattern of outputs suggests that as $x$ approaches $-\infty$, $f(x)$ approaches 0 and as $x$ approaches $\infty$, $f(x)$ approaches 0. This supports the conjecture.

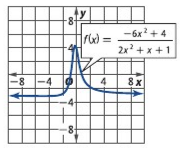
### Guided Practice

**Use the graph of each function to describe its end behavior. Support the conjecture numerically.** **5A–B. See margin.**

**5A.**




**5B.**




Knowing the end behavior of a function can help you solve real-world problems.

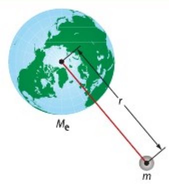
**Real-World Link**

The form $U(r) = -\frac{GmM_e}{r}$ for gravitational potential energy is most useful for calculating the velocity required to escape Earth's gravity, 40,270 km/h.

**Source:** *The Mechanical Universe*

## Real-World Example 6 Apply End Behavior

**PHYSICS Gravitational potential energy of an object is given by $U(r) = -\frac{GmM_e}{r}$, where $G$ is Newton's gravitational constant, $m$ is the mass of the object, $M_e$ is the mass of Earth, and $r$ is the distance from the object to the center of Earth as shown. What happens to the gravitational potential energy of the object as it moves farther and farther from Earth?**




We are asked to describe the end behavior $U(r)$ for large values of $r$. That is, we are asked to find $\lim_{r \to \infty} U(r)$. Because $G$, $m$, and $M_e$ are constant values, the product $GmM_e$ is also a constant value. For increasing values of $r$, the fraction $-\frac{GmM_e}{r}$ will approach 0, so $\lim_{r \to \infty} U(r) = 0$. Therefore, as an object moves farther from Earth, its gravitational potential energy approaches 0.

### Guided Practice

6. PHYSICS Dynamic pressure is the pressure generated by the velocity of the moving fluid and is given by $q(v) = \frac{\rho v^2}{2}$, where $\rho$ is the density of the fluid and $v$ is the velocity of the fluid. What would happen to the dynamic pressure of a fluid if the velocity were to continuously increase? **The dynamic pressure would approach ∞.**



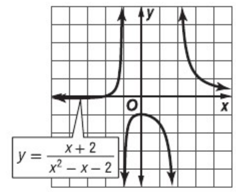


## Differentiated Instruction AL

**Visual/Spatial Learners** Have students work in small groups to create a grid on a large piece of paper. Have the students make tick marks on the axes from −50 to 50. Ask students to choose a function from the lesson that is discontinuous and plot points at every fifth interval on the $x$-axis. Have them choose a second function, one with finite end behavior, and plot a second set of points. Have students describe the discontinuity and end behavior with their large graphs.

## Additional Examples

**5** Use the graph of $f(x) = \frac{x + 2}{x^2 - x - 2}$ to describe its end behavior. Support the conjecture numerically.




From the graph, it appears that as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to 0$.

| $x$ | $f(x)$ |
|---|---|
| −10,000 | $-1 \times 10^{-6}$ |
| −1000 | $-1 \times 10^{-5}$ |
| 0 | −1 |
| 1000 | $1 \times 10^{-3}$ |
| 10,000 | $1 \times 10^{-4}$ |

**6** **PHYSICS** The symmetric energy function is $E = \frac{x^2 + y^2}{2}$. If the $y$-value is held constant, what happens to the value of symmetric energy when the $x$-value approaches $-\infty$?

$\lim_{x \to -\infty} E = \infty$

### Guided Practice

**5a.** From the graph, it appears that as $x \to -\infty$, $f(x) \to 3$ and as $x \to \infty$, $f(x) \to 3$.

| $x$ | $f(x)$ |
|---|---|
| −10,000 | 3.0005 |
| −1000 | 3.005 |
| 0 | −2 |
| 1000 | 2.995 |
| 10,000 | 2.9995 |

**5b.** From the graph, it appears that as $x \to -\infty$, $f(x) \to -3$ and as $x \to \infty$, $f(x) \to -3$.

| $x$ | $f(x)$ |
|---|---|
| −10,000 | −3.0001 |
| −1000 | −3.001 |
| 0 | 4 |
| 1000 | −2.998 |
| 10,000 | −2.9998 |

# 3 Practice

### Formative Assessment

Use Exercises 1–41 to check for understanding.

Then use the table below to customize your assignments for students.

### Additional Answers

1. Continuous; $f(-5) = \sqrt{21}$ or about 4.58, $\lim_{x \to -5} f(x) \approx 4.58$, and $\lim_{x \to -5} f(x) = f(-5)$.
2. Continuous; $f(8) = \sqrt{13} \approx 3.606$, $\lim_{x \to 8} f(x) \approx 3.606$, and $\lim_{x \to 8} f(x) = f(8)$.
3. Discontinuous at $x = -6$; $h(-6)$ is undefined and $\lim_{x \to -6} 6(x) = -12$, so $h(x)$ has a removable discontinuity at $x = -6$. Continuous at $x = 6$. $h(6) = 0$, $\lim_{x \to 6} h(x) = 0$, and $\lim_{x \to 6} h(x) = h(6)$.
4. Discontinuous at $x = -5$; $h(-5)$ is undefined and $\lim_{x \to -5} h(x) = -10$, so $h(x)$ has a removable discontinuity at $x = -5$. Continuous at $x = 5$. $h(5) = 0$, $\lim_{x \to 5} h(x) = 0$, and $\lim_{x \to 5} h(x) = h(5)$.
5. Discontinuous; $g(1)$ is undefined and $g(x)$ approaches $-\infty$ as $x$ approaches 1 from the left and $\infty$ as $x$ approaches 1 from the right, so $g(x)$ has an infinite discontinuity at $x = 1$.
6. Discontinuous at $x = -2$; $g(-2)$ is undefined and $g(x)$ approaches $-\infty$ as $x$ approaches $-2$ from the left and $\infty$ as $x$ approaches $-2$ from the right, so $g(x)$ has an infinite discontinuity at $x = -2$. Continuous at $x = 2$; $g(2) = 0$, $\lim_{x \to 2} g(x) = 0$, and $\lim_{x \to 2} g(x) = g(2)$.

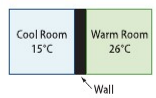
## Exercises

**Determine whether each function is continuous at the given $x$-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite, jump,* or *removable*.** (Examples 1 and 2) **1–6. See margin.**

1. $f(x) = \sqrt{x^2 - 4}$; at $x = -5$
2. $f(x) = \sqrt{x + 5}$; at $x = 8$
3. $h(x) = \frac{x^2 - 36}{x + 6}$; at $x = -6$ and $x = 6$
4. $h(x) = \frac{x^2 - 25}{x + 5}$; at $x = -5$ and $x = 5$
5. $g(x) = \frac{x}{x - 1}$; at $x = 1$
6. $g(x) = \frac{2 - x}{2 + x}$; at $x = -2$ and $x = 2$
7. $h(x) = \frac{x - 4}{x^2 - 5x + 4}$; at $x = 1$ and $x = 4$
8. $h(x) = \frac{x(x - 6)}{x^3}$; at $x = 0$ and $x = 6$
9. $f(x) = \begin{cases} 4x - 1 & \text{if } x \le -6 \\ -x + 2 & \text{if } x > -6 \end{cases}$; at $x = -6$
10. $f(x) = \begin{cases} x^2 - 1 & \text{if } x > -2 \\ x - 5 & \text{if } x \le -2 \end{cases}$; at $x = -2$

**7–10. See Chapter 11 Answer Appendix.**

11. **PHYSICS** A wall separates two rooms with different temperatures. The heat transfer in watts between the two rooms can be modeled by $f(w) = \frac{7.4}{w}$, where $w$ is the wall thickness in meters. (Examples 1 and 2) **a–c. See Chapter 11 Answer Appendix.**




a. Determine whether the function is continuous at $w = 0.4$. Justify your answer using the continuity test.

b. Is the function continuous? Justify your answer using the continuity test. If discontinuous, identify the type of discontinuity as *infinite, jump,* or *removable*.

c. Graph the function to verify your conclusion from part **b**.

**12a–c. See Chapter 11 Answer Appendix.**

12. **CHEMISTRY** A solution must be diluted so it can be used in an experiment. Adding a 4-molar solution to a 10-molar solution will decrease the concentration. The concentration $C$ of the mixture can be modeled by $C(x) = \frac{500 + 4x}{50 + x}$, where $x$ is the number of liters of 4-molar solution added. (Examples 1 and 2)

a. Determine whether the function is continuous at $x = 10$. Justify the answer using the continuity test.

b. Is the function continuous? Justify your answer using the continuity test. If discontinuous, identify the type of discontinuity as *infinite, jump,* or *removable* and describe what affect, if any, the discontinuity has on the concentration of the mixture.

c. Graph the function to verify your conclusion from part **b**.

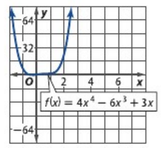
**Determine between which consecutive integers the real zeros of each function are located on the given interval.** (Example 3)

13. $f(x) = x^3 - x^2 - 3$; $[-2, 4]$ **1 and 2**
14. $g(x) = -x^3 + 6x + 2$; $[-4, 4]$ **−3 and −2, −1 and 0, 2 and 3**
15. $f(x) = 2x^4 - 3x^3 + x^2 - 3$; $[-3, 3]$ **−1 and 0, 1 and 2**
16. $h(x) = -x^4 + 4x^3 - 5x - 6$; $[3, 5]$ **3 and 4**
17. $f(x) = 3x^3 - 6x^2 - 2x + 2$; $[-2, 4]$ **−1 and 0, 0 and 1, 2 and 3**
18. $g(x) = \frac{x^2 + 3x - 3}{x^2 + 1}$; $[-4, 3]$ **−4 and −3, 0 and 1**
19. $h(x) = \frac{x^2 + 4}{x - 5}$; $[-2, 4]$ **no zeros on the interval**
20. $f(x) = \sqrt{x^2 - 6} - 6$; $[3, 8]$ **6 and 7**
21. $g(x) = \sqrt{x^3 + 1} - 5$; $[0, 5]$ **2 and 3**

**22–29. See Chapter 11 Answer Appendix.**

**Use the graph of each function to describe its end behavior. Support the conjecture numerically.** (Examples 4 and 5)

22.

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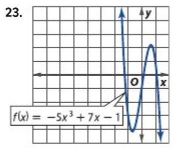


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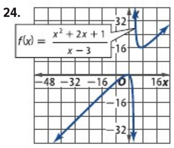


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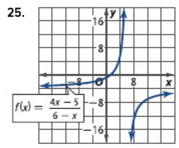


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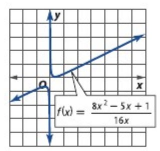


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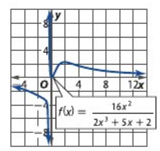


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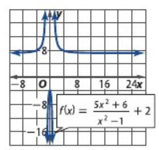
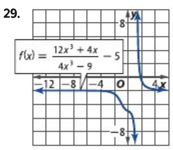



29.



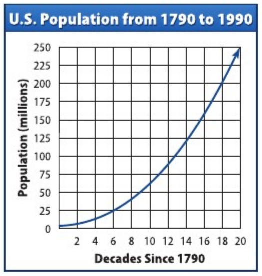





## Differentiated Homework Options (OL) (BL) (AL)

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–41, 58–60, 62–83 | 1–41 odd, 80–82 | 2–40 even, 58–60, 62–79 |
| OL On Level | 1–43 odd, 44, 45–49 odd, 50, 51, 53, 55–60, 62–83 | 1–41, 80–83 | 42–60, 62–79 |
| BL Beyond Level | 42–83 | | |

**30.** **POPULATION** The U.S. population from 1790 to 1990 can be modeled by $p(x) = 0.0057x^3 + 0.4895x^2 + 0.3236x + 3.8431$, where $x$ is the number of decades after 1790. Use the end behavior of the graph to describe the population trend. Support the conjecture numerically. Does this trend seem realistic? Explain your reasoning. (Example 4) **See margin.**

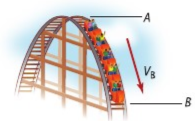



**31.** **CHEMISTRY** A catalyst is used to increase the rate of a chemical reaction. The reaction rate $R$, or the speed at which the reaction is occurring, is given by $R(x) = \frac{0.5x}{x + 12}$, where $x$ is the concentration of the solution in milligrams of solute per liter of solution. (Example 5) **See margin.**

a. Graph the function using a graphing calculator.

b. What does the end behavior of the graph mean in the context of this experiment? Support the conjecture numerically.

**32.** **ROLLER COASTERS** The speed of a roller coaster after it drops from a height $A$ to a height $B$ is given by $f(h_A) = \sqrt{2g(h_A - h_B)}$, where $h_A$ is the height at point $A$, $h_B$ is the height at point $B$, and $g$ is the acceleration due to gravity. What happens to $f(h_A)$ as $h_B$ decreases to 0? (Example 6) **See margin.**




**Use logical reasoning to determine the end behavior or limit of the function as $x$ approaches infinity. Explain your reasoning.** (Example 6) **33–40. See margin.**

**33.** $q(x) = -\frac{24}{x}$

**34.** $f(x) = \frac{0.8}{x^2}$

**35.** $p(x) = \frac{x + 1}{x - 2}$

**36.** $m(x) = \frac{4 + x}{2x + 6}$

**37.** $c(x) = \frac{5x^2}{x^3 + 2x + 1}$

**38.** $k(x) = \frac{4x^2 - 3x - 1}{11x}$

**39.** $h(x) = 2x^5 + 7x^3 + 5$

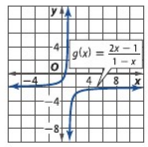
**40.** $g(x) = x^4 - 9x^2 + \frac{x}{4}$



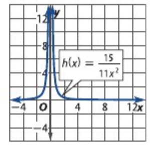
**41.** **PHYSICS** The kinetic energy of an object in motion can be expressed as $E(m) = \frac{p^2}{2m}$, where $p$ is the momentum and $m$ is the mass of the object. If sand is added to a moving railway car, what would happen as $m$ continues to increase? (Example 6) **See margin.**

**Use each graph to determine the $x$-value(s) at which each function is discontinuous. Identify the type of discontinuity. Then use the graph to describe its end behavior. Justify your answers.** **42–43. See Chapter 11 Answer Appendix.**

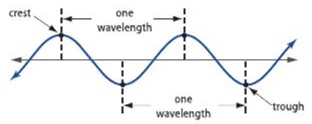
**42.**




**43.**




**44.** **PHYSICS** The wavelength $\lambda$ of a periodic wave is the distance between consecutive corresponding points on the wave, such as two crests or troughs.




The frequency $f$, or number of wave crests that pass any given point during a given period of time, is given by $f(\lambda) = \frac{c}{\lambda}$, where $c$ is the speed of light or $2.99 \cdot 10^8$ meters per second. **a–b. See Chapter 11 Answer Appendix.**

a. Graph the function using a graphing calculator.

b. Use the graph to describe the end behavior of the function. Support your conjecture numerically.

c. Is the function continuous? If not, identify and describe any points of discontinuity.
**No; there is an infinite discontinuity at $\lambda = 0$.**

**GRAPHING CALCULATOR** **Graph each function and determine whether it is continuous. If discontinuous, identify and describe any points of discontinuity. Then describe its end behavior and locate any zeros.** **45–49. See Chapter 11 Answer Appendix.**

**45.** $f(x) = \frac{x^2}{x^3 - 4x^2 + x + 6}$

**46.** $g(x) = \frac{x^2 - 9}{x^3 - 5x^2 - 18x + 72}$

**47.** $h(x) = \frac{4x^2 + 11x - 3}{x^2 + 3x - 18}$

**48.** $h(x) = \frac{x^3 - 4x^2 - 29x - 24}{x^2 - 2x - 15}$

**49.** $h(x) = \frac{x^3 - 5x^2 - 26x + 120}{x^2 + x - 12}$

727

**WatchOut!**

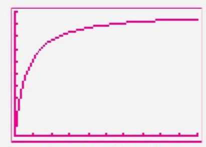
**Common Error** For Exercises 45–49, students may forget to use parentheses around each polynomial when entering the function into their calculators. Remind them that parentheses are needed for the calculator to correctly graph the function.

## Additional Answers

**30.** From the graph, it appears that as the number of decades since 1790 increases without bound, the population increases without bound. This trend does not seem realistic because there is no way to guarantee that the U.S. population will increase without bound in the future.

| $x$ | 0 | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | 3.8 | 61.7 | $1.1 \cdot 10^4$ | $6.2 \cdot 10^6$ | $5.8 \cdot 10^9$ |

**31a.**



[0, 200] scl: 20 by [0, 0.5] scl: 0.05

**31b.** Sample answer: The end behavior of the graph indicates that as the concentration of the catalyst solution is increased, the chemical reaction rate approaches 0.5.

| $x$ | 0 | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $R(x)$ | 0 | 0.2273 | 0.4464 | 0.4941 | 0.4994 |

**32.** Sample answer: As height $h_B$ decreases, the speed at point $B$ approaches $\sqrt{2gh_A}$.

**33.** Sample answer: As $x \to \infty$, the fraction will decrease, and $q(x)$ will approach 0.

**34.** Sample answer: As $x \to \infty$, the fraction will decrease, and $f(x)$ will approach 0.

**35.** Sample answer: As $x \to \infty$, the fraction will approach $\frac{x}{x}$, so $p(x)$ will approach 1.

**36.** Sample answer: As $x \to \infty$, the fraction will approach $\frac{x}{2x}$, so $m(x)$ will approach $\frac{1}{2}$.

**37.** Sample answer: As $x \to \infty$, the fraction will decrease, and $c(x)$ will approach 0.

**38.** Sample answer: As $x \to \infty$, the numerator of the fraction will become very large compared to the denominator, so $k(x)$ will approach $\infty$.

**39.** Sample answer: As $x \to \infty$, $h(x)$ will increase, without bound and approach $\infty$.

**40.** Sample answer: As $x \to \infty$, $g(x)$ will increase without bound and approach $\infty$.

**41.** Sample answer: As the mass of the railway car continues to increase, the railway car's kinetic energy will approach 0.

## Additional Answers

**58.** Removable; $f(0)$ is undefined and $\lim\limits_{x \to 0} f(x) = 1$.

**59.** Infinite; $f(0)$ is undefined, and $f(x)$ approaches $-\infty$ as $x$ approaches 0 from the left and $\infty$ as $x$ approaches 0 from the right.

## Additional Answers

**60.** Saeed; sample answer: The relation in the graph is not a function because there are two $y$-values paired with the same $x$-value.

**62.** $\lim\limits_{x \to -\infty} f(x) = -\infty$; because $f$ is even, $f(-x) = f(x)$.

**63.** $\lim\limits_{x \to -\infty} f(x) = \infty$; because $f$ is odd, $f(-x) = -f(x)$.

**64.** $\lim\limits_{x \to -\infty} f(x) = -\infty$; because the graph of $f$ is symmetric with respect to the origin, $f(-x) = -f(x)$.

**65.** $\lim\limits_{x \to -\infty} f(x) = \infty$; with $y$-axis symmetry, $f(x) = f(-x)$.

**50.** **VEHICLES** The number $A$ of alternative-fueled vehicles in use in the United States from 2000 to 2010 can be approximated by $f(t) = 5420t^2 - 14{,}726t + 531{,}750$, where $t$ represents the number of years since 2000.

a. Graph the function. **See Chapter 11 Answer Appendix.**

b. About how many alternative-fueled vehicles were there in the United States in 2008? **760,822 vehicles**

c. As time goes by, what will the number of alternative-fueled vehicles approach, according to the model? Do you think that the model is valid after 2010? Explain. **See Chapter 11 Answer Appendix.**

**GRAPHING CALCULATOR Graph each function, and describe its end behavior. Support the conjecture numerically, and provide an effective viewing window for each graph.** **51–54. See Chapter 11 Answer Appendix.**

**51.** $f(x) = -x^4 + 12x^3 + 4x^2 - 4$

**52.** $g(x) = x^5 - 20x^4 + 2x^3 - 5$

**53.** $f(x) = \dfrac{16x^2}{x^2 + 15x}$

**54.** $g(x) = \dfrac{8x - 24x^3}{14 + 2x^3}$

**55a–c. See Chapter 11 Answer Appendix.**

**55.** **BUSINESS** Faris is starting a small business screen-printing and selling mugs. Each mugs costs AED 3 to produce. He initially invested AED 4000 for a screen printer and other business needs.

a. Write a function to represent the average cost per mug as a function of the number of mugs sold $n$.

b. Use a graphing calculator to graph the function.

c. As the number of mugs sold increases, what value does the average cost approach?

**56.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate limits. Consider $f(x) = \dfrac{ax^3 + b}{cx^3 + d}$, where $a$ and $c$ are nonzero integers, and $b$ and $d$ are integers.

a. **TABULAR** Let $c = 1$, and choose three different sets of values for $a$, $b$, and $d$. Write the function with each set of values. Copy and complete the table below. **a–c. See Chapter 11 Answer Appendix.**

| $c = 1$ | | | | |
|---|---|---|---|---|
| $a$ | $b$ | $d$ | $\lim\limits_{x \to \infty} f(x)$ | $\lim\limits_{x \to -\infty} f(x)$ |
| 3 | 2 | 4 | 3 | 3 |
| −1 | 5 | 7 | −1 | −1 |
| 9 | −6 | 8 | 9 | 9 |

b. **TABULAR** Choose three different sets of values for each variable: one set with $a > c$, one set with $a < c$, and one set with $a = c$. Write each function, and create a table as you did in part **a**.

c. **ANALYTICAL** Make a conjecture about the limit of $f(x) = \dfrac{ax^3 + b}{cx^3 + d}$ as $x$ approaches positive and negative infinity.

**57.** Graph several different functions of the form $f(x) = x^n + ax^{n-1} + bx^{n-2}$, where $n$, $a$, and $b$ are integers, $n \geq 2$.

a. Make a conjecture about the end behavior of the function when $n$ is positive and even. Include at least one graph to support your conjecture.

b. Make a conjecture about the end behavior of the function when $n$ is positive and odd. Include at least one graph to support your conjecture.

**57a–b. See Chapter 11 Answer Appendix.**

### H.O.T. Problems Use Higher-Order Thinking Skills

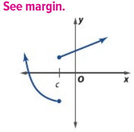
**REASONING Determine whether each function has an *infinite*, *jump*, or *removable* discontinuity at $x = 0$. Explain.**

**58.** $f(x) = \dfrac{x^5 + x^6}{x^5}$

**59.** $f(x) = \dfrac{x^4}{x^5}$

**58–59. See margin.**

**60.** **ERROR ANALYSIS** Khalid and Saeed are determining whether the relation graphed below is continuous at point $c$. Khalid thinks that it is the graph of a function $f(x)$ that is discontinuous at point $c$ because $\lim\limits_{x \to c} f(x) = f(c)$ from only one side of $c$. Saeed thinks that the graph is not a function because when $x = c$, the relation has two different $y$-values. Is either of them correct? Explain your reasoning. **See margin.**




**61.** **CHALLENGE** Determine the values of $a$ and $b$ so that $f$ is continuous.

$$f(x) = \begin{cases} x^2 + a & \text{if } x \geq 3 \\ bx + a & \text{if } -3 < x < 3 \\ \sqrt{-b - x} & \text{if } x \leq -3 \end{cases}$$

**$a = 9$, $b = 3$**

**REASONING Find $\lim\limits_{x \to -\infty} f(x)$ for each of the following. Explain your reasoning.** **62–65. See margin.**

**62.** $\lim\limits_{x \to \infty} f(x) = -\infty$ and $f$ is an even function.

**63.** $\lim\limits_{x \to \infty} f(x) = -\infty$ and $f$ is an odd function.

**64.** $\lim\limits_{x \to \infty} f(x) = \infty$ and the graph of $f$ is symmetric with respect to the origin.

**65.** $\lim\limits_{x \to \infty} f(x) = \infty$ and the graph of $f$ is symmetric with respect to the $y$-axis.

**See Chapter 11 Answer Appendix.**

**66.** **WRITING IN MATH** Provide an example of a function with a removable discontinuity. Explain how this discontinuity can be eliminated. How does eliminating the discontinuity affect the function?





728 | Lesson 11-3 | Continuity, End Behavior, and Limits

## Spiral Review

**GRAPHING CALCULATOR** **Graph each function. Analyze the graph to determine whether each function is *even, odd,* or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.** (Lesson 11-2) **67–69. See Chapter 11 Answer Appendix.**

**67.** $h(x) = \sqrt{x^2 - 16}$ **68.** $f(x) = \frac{2x + 1}{x}$ **69.** $g(x) = x^5 - 5x^3 + x$

**State the domain of each function.** (Lesson 11-1)

**70.** $f(x) = \frac{4x + 6}{x^2 + 3x + 2}$ $D = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

**71.** $g(x) = \frac{x + 3}{x^2 - 2x - 10}$ $D = (-\infty, 1 - \sqrt{11}) \cup (1 - \sqrt{11}, 1 + \sqrt{11}) \cup (1 + \sqrt{11}, \infty)$

**72.** $g(a) = \sqrt{2 - a^2}$ $D = [-\sqrt{2}, \sqrt{2}]$

**73.** **POSTAL SERVICE** The U.S. Postal Service uses five-digit ZIP codes to route letters and packages to their destinations.

a. How many ZIP codes are possible if the numbers 0 through 9 are used for each of the five digits? **100,000**

b. Suppose that when the first digit is 0, the second, third, and fourth digits cannot be 0. How many five-digit ZIP codes are possible if the first digit is 0? **7290**

c. In 1983, the U.S. Postal Service introduced the ZIP + 4, which added four more digits to the existing five-digit ZIP codes. Using the numbers 0 through 9, how many additional ZIP codes were possible? **999,900,000**

**Given** $A = \begin{bmatrix} -4 & 10 & -2 \\ 3 & -3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 8 & -5 & 4 \\ 4 & 9 & -3 \end{bmatrix}$**, solve each equation for** $X$.

**74.** $3X - B = A$ **74.** $\begin{bmatrix} \frac{4}{3} & \frac{5}{3} & \frac{2}{3} \\ \frac{7}{3} & 2 & -\frac{2}{3} \end{bmatrix}$

**75.** $2B + X = 4A$ $\begin{bmatrix} -32 & 50 & -16 \\ 4 & -30 & 10 \end{bmatrix}$

**76.** $A - 5X = B$ $\begin{bmatrix} -\frac{12}{5} & 3 & -\frac{6}{5} \\ -\frac{1}{5} & -\frac{12}{5} & \frac{4}{5} \end{bmatrix}$

**Solve each system of equations.**

**77.** $4x - 6y + 4z = 12$
$6x - 9y + 6z = 18$
$5x - 8y + 10z = 20$ **infinite solutions**

**78.** $x + 2y + z = 10$
$2x - y + 3z = -5$
$2x - 3y - 5z = 27$ **(7, 4, −5)**

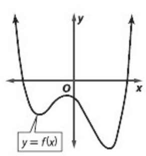
**79.** $2x - y + 3z = -2$
$x + 4y - 2z = 16$
$5x + y - z = 14$ **(2, 3, −1)**

## Skills Review for Standardized Tests

**80.** **SAT/ACT** At Sheikh Zayed Secondary School, 36 students are taking either calculus or physics or both, and 10 students are taking both calculus and physics. If there are 31 students in the calculus class, how many students are there in the physics class? **D**

A 5 C 11 E 21
B 8 D 15

**81.** Which of the following statements could be used to describe the end behavior of $f(x)$? **J**




F $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$

G $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

H $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$

J $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$

**82.** **REVIEW** Suha's locker code includes three numbers between 1 and 45, inclusive. None of the numbers can repeat. How many possible locker permutations are there? **85,140**

**83.** **REVIEW** Suppose a figure consists of three concentric circles with radii of 1 m, 2 m, and 3 m. Find the probability that a point chosen at random lies in the outermost region (between the second and third circles). **D**

A $\frac{1}{3}$ C $\frac{4}{9}$
B $\frac{\pi}{9}$ D $\frac{5}{9}$





### WatchOut!

**Common Error** Exercise 56 part **c** may confuse some students with what appears to be many variables. Remind students that $x$ is the only independent variable and that $a$, $b$, $c$, and $d$ are all constants.

**Error Analysis** For Exercise 60, students should use the three steps for determining continuity. The value $f(c)$ does exist (Step 1). However, the value of $f(c)$ is different as $c$ is approached from the left and $c$ is approached from the right. Saeed is correct that the graph of the relation does not pass the vertical line test. There are two $y$-values when $x$ is $c$. The graph does not represent a function.

## 4 Assess

**Yesterday's News** Ask students to describe how analyzing graphs of relations and functions helped them understand continuity and end behavior.

### DifferentiatedInstruction BL

**Extension** Find values of $m$ and $b$ so that

$$f(x) = \begin{cases} -x^2 \text{ if } x \le 0 \\ mx + b \text{ if } 0 < x < 1 \\ x^2 \text{ if } x \ge 1 \end{cases}$$

is continuous. Sample answer: $m = 1$, $b = 0$

# 1 Focus

## VerticalAlignment

**Before Lesson 11-4** Find function values.

**Lesson 11-4** Determine intervals on which functions are increasing, constant, or decreasing, and determine maxima and minima of functions.
Determine the average rate of change of a function.

**After Lesson 11-4** Graph and describe parent functions.

# 2 Teach

## Scaffolding Questions

Have students read the **Why?** section of the lesson.

**Ask:**

- A business owner improves manufacturing processes after a steady drop in profit. The changes take place over June, July, and August. When should the graph of profit change from decreasing to increasing? Sample answer: during or shortly after June, July, or August

# LESSON 11-4 Extrema and Average Rates of Change

**Then**

- You found function values. (Lesson 11-1)

**Now**

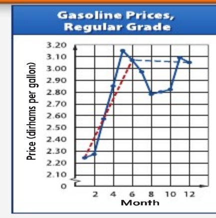
1. Determine intervals on which functions are increasing, constant, or decreasing, and determine maxima and minima of functions.
2. Determine the average rate of change of a function.

**Why?**

- The graph shows the average price of regular-grade gasoline in the U.S. from January to December.

The highest average price was about AED 3.15 per gallon in May.

The slopes of the red and blue dashed lines show that the price of gasoline changed more rapidly in the first half of the year than in the second half.




## NewVocabulary

increasing
decreasing
constant
critical point
extrema
maximum
minimum
point of inflection
average rate of change
secant line

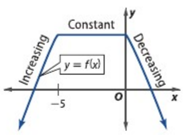
## 1 Increasing and Decreasing Behavior

An analysis of a function can also include a description of the intervals on which the function is increasing, decreasing, or constant.

Consider the graph of $f(x)$ shown. As you move from *left to right*, $f(x)$ is

- increasing or *rising* on $(-\infty, -5)$,
- constant or *flat* on $(-5, 0)$, and
- decreasing or *falling* on $(0, \infty)$.

These graphical interpretations can also be described algebraically.

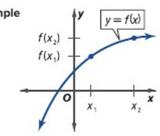



### KeyConcept Increasing, Decreasing, and Constant Functions

**Words** A function $f$ is **increasing** on an interval $I$ if and only if for any two points in $I$, a positive change in $x$ results in a positive change in $f(x)$.

**Symbols** For every $x_1$ and $x_2$ in an interval $I$, $f(x_1) < f(x_2)$ when $x_1 < x_2$.

**Example**

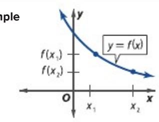



Interval: $(-\infty, \infty)$

**Words** A function $f$ is **decreasing** on an interval $I$ if and only if for any two points in $I$, a positive change in $x$ results in a negative change in $f(x)$.

**Symbols** For every $x_1$ and $x_2$ in an interval $I$, $f(x_1) > f(x_2)$ when $x_1 < x_2$.

**Example**

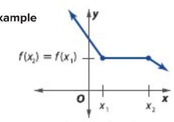



Interval: $(-\infty, \infty)$

**Words** A function $f$ is **constant** on an interval $I$ if and only if for any two points in $I$, a positive change in $x$ results in a zero change in $f(x)$.

**Symbols** For every $x_1$ and $x_2$ in an interval $I$, $f(x_1) = f(x_2)$ when $x_1 < x_2$.

**Example**




Interval: $(a, b)$

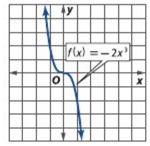
**Example 1** Analyze Increasing and Decreasing Behavior

**Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Support the answer numerically.**

**WatchOut!**

**Intervals** A function is neither increasing nor decreasing at a point, so the symbols ( and ) should be used when describing the intervals on which a function is increasing or decreasing.

**a.** $f(x) = -2x^3$




**Analyze Graphically**

When viewed from left to right, the graph of $f$ falls for all real values of $x$. Therefore, we can conjecture that $f$ is decreasing on $(-\infty, \infty)$.

**Support Numerically**

Create a table using values in the interval.

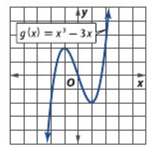
| x | −8 | −6 | −4 | −2 | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| f(x) | 1024 | 432 | 128 | 16 | 0 | −16 | −128 | −432 | −1024 |

The table shows that as $x$ increases, $f(x)$ decreases. This supports the conjecture.

**StudyTip**

**Increasing, Decreasing, and Constant Functions** Functions that increase, decrease, or are constant for all $x$ in their domain are called *increasing, decreasing,* or *constant functions,* respectively. The function in Example 1a is a decreasing function, while the function in Example 1b cannot be classified as increasing or decreasing because it has an interval where it is increasing and another interval where it is decreasing.

**b.** $g(x) = x^3 - 3x$




**Analyze Graphically**

From the graph, we can estimate that $f$ is increasing on $(-\infty, -1)$, decreasing on $(-1, 1)$, and increasing on $(1, \infty)$.

**Support Numerically**

Create a table of values using $x$-values in each interval.

$(-\infty, -1)$:

| x | −13 | −11 | −9 | −7 | −5 | −3 |
|---|---|---|---|---|---|---|
| f(x) | −2158 | −1298 | −702 | −322 | −110 | −18 |

$(-1, 1)$:

| x | −0.75 | −0.5 | 0 | 0.5 | 0.75 |
|---|---|---|---|---|---|
| f(x) | 1.828 | 1.375 | 0 | −1.375 | −1.828 |

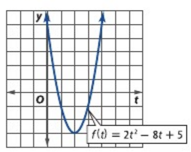
$(1, \infty)$:

| x | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|
| f(x) | 18 | 110 | 322 | 702 | 1298 | 2158 |

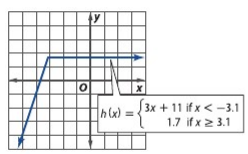
The tables show that as $x$ increases to $-1$, $f(x)$ increases; as $x$ increases from $-1$ to 1, $f(x)$ decreases; as $x$ increases from 1, $f(x)$ increases. This supports the conjecture.

**GuidedPractice** **1A–B. See Margin.**

1A.




1B.




While a graphical approach to identify the intervals on which a function is increasing, decreasing, or constant can be supported numerically, calculus is often needed to confirm this behavior and to confirm that a function does not change its behavior beyond the domain shown.





## Additional Answers (Guided Practice)

**1A.** $f$ is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

$(-\infty, 2)$

| x | −10 | −8 | −6 | −4 | −2 | 0 |
|---|---|---|---|---|---|---|
| f(x) | 285 | 197 | 125 | 69 | 29 | 5 |

$(2, \infty)$

| x | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|
| f(x) | 5 | 29 | 69 | 125 | 197 | 285 |

**1B.** $h$ is increasing on $(-\infty, -3)$ and constant on $(-3, \infty)$.

$(-\infty, -3)$

| x | −9 | −8 | −7 | −6 | −5 | −4 |
|---|---|---|---|---|---|---|
| f(x) | −16 | −13 | −10 | −7 | −4 | −1 |

$(-3, \infty)$

| x | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| f(x) | 1.7 | 1.7 | 1.7 | 1.7 | 1.7 | 1.7 |

- A politician's monthly approval ratings have been up and down over the last year. How could he find the average rate of change over two months? He could subtract the first month's rating from the second month's rating and divide by 2.

## 1 Increasing and Decreasing Behavior

**Example 1** shows how to find intervals on which a function is increasing, decreasing, or constant. **Examples 2–4** show how to find and use extrema.

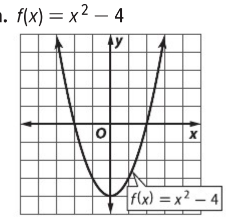
### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

### Additional Example

**1** Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Support the answer numerically.

**a.** $f(x) = x^2 - 4$




$f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

$(-\infty, 0)$

| x | −20 | −15 | −10 | −5 |
|---|---|---|---|---|
| f(x) | 396 | 221 | 96 | 21 |

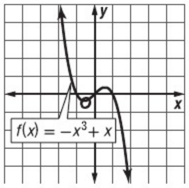
$(0, \infty)$

| x | 5 | 10 | 15 | 20 |
|---|---|---|---|---|
| f(x) | 21 | 96 | 221 | 396 |

*(continued on the next page)*

## Additional Example

**b.** $f(x) = -x^3 + x$




$f$ is decreasing on $\left(-\infty, -\frac{1}{2}\right)$, increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, and decreasing on $\left(\frac{1}{2}, \infty\right)$.

$\left(-\infty, -\frac{1}{2}\right)$

| $x$ | −10 | −8 | −6 | −4 | −2 |
|---|---|---|---|---|---|
| $f(x)$ | 990 | 504 | 210 | 60 | 6 |

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

| $x$ | −0.4 | −0.2 | 0 | 0.2 | 0.4 |
|---|---|---|---|---|---|
| $f(x)$ | −0.34 | −0.19 | 0 | 0.19 | 0.34 |

$\left(\frac{1}{2}, \infty\right)$

| $x$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $f(x)$ | −6 | −60 | −210 | −504 | −990 |

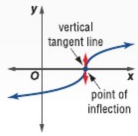
## Focus on Mathematical Content

**Relative and Absolute Extrema**

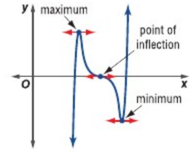
Relative minima or maxima relate to an interval, and there can be multiple relative minima or maxima for a function. Absolute minima and maxima are not limited to an interval. They are the extrema for the entire domain of the function and there is at most one absolute minimum and one absolute maximum for a function.

### StudyTip

**Tangent Line** Recall from geometry that a line is tangent to a curve if it intersects a curve in exactly one point.




**Critical points** of a function are those points at which a line drawn tangent to the curve is horizontal or vertical. **Extrema** are critical points at which a function changes its increasing or decreasing behavior. At these points, the function has a **maximum** or a **minimum** value, either relative or absolute. A **point of inflection** can also be a critical point. At these points, the graph changes its shape, but not its increasing or decreasing behavior. Instead, the curve changes from being bent upward to being bent downward, or vice versa.




### KeyConcept Relative and Absolute Extrema

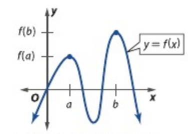
**Words** A *relative maximum* of a function $f$ is the greatest value $f(x)$ can attain on some interval of the domain.

**Symbols** $f(a)$ is a relative maximum of $f$ if there exists an interval $(x_1, x_2)$ containing $a$ such that $f(a) > f(x)$ for every $x \neq a$ in $(x_1, x_2)$.

**Words** If a relative maximum is the greatest value a function $f$ can attain over its entire domain, then it is the *absolute maximum*.

**Symbols** $f(b)$ is the absolute maximum of $f$ if $f(b) > f(x)$ for every $x \neq b$, in the domain of $f$.

**Model**




$f(a)$ is a relative maximum of $f$.
$f(b)$ is the absolute maximum of $f$.

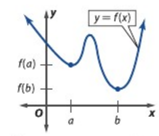
**Words** A *relative minimum* of a function $f$ is the least value $f(x)$ can attain on some interval of the domain.

**Symbols** $f(a)$ is a relative minimum of $f$ if there exists an interval $(x_1, x_2)$ containing $a$ such that $f(a) < f(x)$ for every $x \neq a$ in $(x_1, x_2)$.

**Words** If a relative minimum is the least value a function $f$ can attain over its entire domain, then it is the *absolute minimum*.

**Symbols** $f(b)$ is the absolute minimum of $f$ if $f(b) < f(x)$ for every $x \neq b$, in the domain of $f$.

**Model**



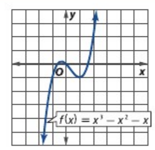

$f(a)$ is a relative minimum of $f$.
$f(b)$ is the absolute minimum of $f$.

### ReadingMath

**Plural Forms** Using Latin, *maxima* is the plural form of *maximum*, *minima* is the plural form of *minimum*, and *extrema* is the plural form of *extremum*.

### Example 2 Estimate and Identify Extrema of a Function

**Estimate and classify the extrema for the graph of $f(x)$. Support the answers numerically.**




**Analyze Graphically**

It appears that $f(x)$ has a relative maximum at $x = -0.5$ and a relative minimum at $x = 1$. It also appears that $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$, so we conjecture that this function has no absolute extrema.

**Support Numerically**

Choose $x$-values in half unit intervals on either side of the estimated $x$-value for each extremum, as well as one very large and one very small value for $x$.

| $x$ | −100 | −1 | −0.5 | 0 | 0.5 | 1 | 1.5 | 100 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1.0 \cdot 10^6$ | −1.00 | 0.125 | 0 | −0.63 | −1 | −0.38 | $9.9 \cdot 10^5$ |

Because $f(-0.5) > f(-1)$ and $f(-0.5) > f(0)$, there is a relative maximum in the interval $(-1, 0)$ near $-0.5$. The approximate value of this relative maximum is $f(-0.5)$ or about 0.13.

**StudyTip**

**Local Extrema** Relative extrema are also called *local* extrema, and absolute extrema are also called *global* extrema.

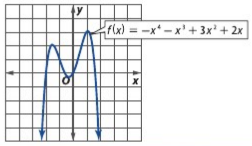
Likewise, because $f(1) < f(0.5)$ and $f(1) < f(1.5)$, there is a relative minimum in the interval $(0.5, 1.5)$ near 1. The approximate value of this relative maximum is $f(1)$ or $-1$.

$f(100) > f(-0.5)$ and $f(-100) < f(1)$, which supports our conjecture that $f$ has no absolute extrema.

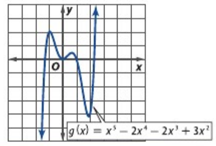
**GuidedPractice**

**Estimate and classify the extrema for the graph of each function. Support the answers numerically.** **2A–B. See margin.**

**2A.**




**2B.**




Because calculus is needed to confirm the increasing and decreasing behavior of a function, calculus is also needed to confirm the relative and absolute extrema of a function. For now, however, you can use a graphing calculator to help you better approximate the location and function value of extrema.

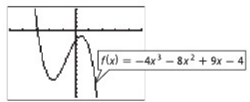
**Example 3** Use a Graphing Calculator to Approximate Extrema

**GRAPHING CALCULATOR** **Approximate to the nearest hundredth the relative or absolute extrema of $f(x) = -4x^3 - 8x^2 + 9x - 4$. State the $x$-value(s) where they occur.**

**TechnologyTip**

**Zooming** When locating maxima and minima, be sure to zoom in or out enough in order to see details and the overall appearance of the graph. The standard window may not tell the entire story.

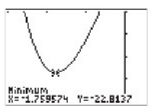
Graph the function and adjust the window as needed so that all of the graph's behavior is visible.



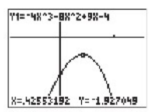

$[-5, 5]$ scl: 1 by $[-30, 10]$ scl: 4

From the graph of $f$, it appears that the function has one relative minimum in the interval $(-2, -1)$ and one relative maximum in the interval $(0, 1)$ of the domain. The end behavior of the graph suggests that this function has no absolute extrema.

Using the minimum and maximum options from the CALC menu of your graphing calculator, you can estimate that $f(x)$ has a relative minimum of $-22.81$ at $x \approx -1.76$ and a relative maximum of $-1.93$ at $x \approx 0.43$.




$[-3, 0.5]$ scl: 1 by $[-28, 12]$ scl: 4




$[-0.9, 1.6]$ scl: 1 by $[-7.3, 2.7]$ scl: 4

**GuidedPractice**

**GRAPHING CALCULATOR** **Approximate to the nearest hundredth the relative or absolute extrema of each function. State the $x$-value(s) where they occur.**

**3A.** $h(x) = 7 - 5x - 6x^2$

**3B.** $g(x) = 2x^3 - 4x^2 - x + 5$

**3A. abs. max: (−0.42, 8.04)**

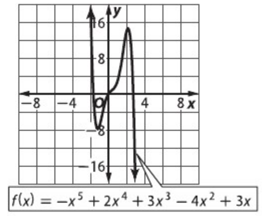
**3B. rel. max: (−0.12, 5.06); rel. min: (1.45, 1.24)**





## Additional Examples

**2** Estimate to the nearest 0.5 unit and classify the extrema for the graph of $f(x)$. Support the answers numerically.




To the nearest 0.5 unit, there is a relative minimum at $x = -1$ and a relative maximum at $x = 2$. There are no absolute extrema.

**3** **GRAPHING CALCULATOR** Approximate to the nearest hundredth the relative or absolute extrema of $f(x) = x^4 - 5x^2 - 2x + 4$. State the $x$-value(s) where they occur. rel. min: (−1.47, 0.80); rel. max: (−0.20, 4.20); abs. min: (1.67, −5.51)

## Teach with Tech

**Spreadsheets** The formula feature of a spreadsheet provides a quick and easy way to produce tables. Have students work in small groups using formulas in spreadsheets to produce tables of values to find relative minima and maxima.

## Additional Answers (Guided Practice)

**2A-B.** Student answers should be close to the approximate extrema values given.

**2A.** rel. max: (−1.52, 2.07); rel. min: (−0.31, −0.31); abs. max: (1.08, 3.04)

**2B.** rel. max: (−0.96, 2.02), (0.66, 0.48); rel. min: (0, 0), (1.90, −4.19)

## Additional Example

**4** **FUEL ECONOMY** Advertisements for a new car claim that a tank of gas will take a driver and three passengers about 360 km. After researching on the Internet, you find the function for km per tank of gas for the car is $f(x) = -0.025x^2 + 3.5x + 240$, where $x$ is the speed of the car in km/h. What speed optimizes the distance the car can travel on a tank of gas? How far will the car travel at that optimum speed? There is a maximum at about 70 km/h. The car will travel 362.5 km when traveling at the optimum speed.

## 2 Average Rate of Change

**Examples 5 and 6** show how to find average rates of change.

### Focus on Mathematical Content

**Average Rate of Change** Finding the average rate of change between two points of a nonlinear function is the same as determining the slope of a line. However, the rate of change between two points of a nonlinear function can change with every pair of points. When calculating the slope of a line, think $\frac{\Delta y}{\Delta x}$. When calculating average rate of change of a function, think $\frac{\Delta f(x)}{\Delta x}$.

*Optimization* is an application of mathematics where one searches for a maximum or a minimum quantity given a set of constraints. If a set of real-world quantities can be modeled by a function, the extrema of the function will indicate these optimal values.

**Real-WorldLink**

Florida produces 95% of the orange crop for orange juice in the United States. In a recent year, more than 800,000 tonnes of oranges were consumed in the United States.

**Source:** U.S. Department of Agriculture

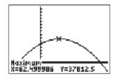
### Real-World Example 4 Use Extrema for Optimization

**AGRICULTURE** **Suppose each of the 75 orange trees in a Florida grove produces 400 oranges per season. Also suppose that for each additional tree planted in the orchard, the yield per tree decreases by 2 oranges. How many additional trees should be planted to achieve the greatest total yield?**

Write a function $P(x)$ to describe the orchard yield as a function of $x$, the number of additional trees planted in the existing orchard.

| orchard yield | = | number of trees in orchard | • | number of oranges produced per tree |
|---|---|---|---|---|
| $P(x)$ | = | $(75 + x)$ | • | $(400 - 2x)$ |

We want to maximize the orchard yield or $P(x)$. Graph this function using a graphing calculator. Then use the maximum option from the CALC menu to approximate the $x$-value that will produce the greatest value for $P(x)$.




[−100, 221.3] scl: 1 by [−12270.5, 87900] scl: 5000

The graph has a maximum of 37,812.5 for $x \approx 62.5$. So by planting an additional 62 trees, the orchard can produce a maximum yield of 37,812 oranges.

**GuidedPractice**

**4.** **CRAFTS** A glass candle holder is in the shape of a right circular cylinder that has a bottom and no top and has a total surface area of $10\pi$ cm$^2$. Determine the radius and the height of the candle holder that will allow the maximum volume. $r \approx 1.83$ cm; $h \approx 1.83$ cm

## 2 Average Rate of Change

In algebra, you learned that the slope between any two points on the graph of a linear function represents a *constant* rate of change. For a nonlinear function, the slope changes between different pairs of points, so we can only talk about the *average* rate of change between any two points.

### KeyConcept Average Rate of Change

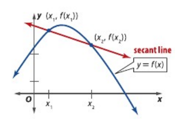
**Words** The **average rate of change** between any two points on the graph of $f$ is the slope of the line through those points.

**Geometry** The line through two points on a curve is called a **secant line**. The slope of the secant line is denoted $m_{sec}$.

**Symbols** The average rate of change on the interval $[x_1, x_2]$ is

$$m_{sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

**Model**




When the average rate of change over an interval is positive, the function increases on average over that interval. When the average rate of change is negative, the function decreases on average over that interval.





734 | Lesson 11-4 | Extrema and Average Rates of Change

## Example 5 Find Average Rates of Change

**Find the average rate of change of $f(x) = -x^3 + 3x$ on each interval.**

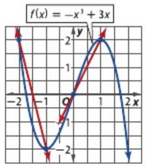


Figure 13.4.1

**a. [−2, −1]**

Use the Slope Formula to find the average rate of change of $f$ on the interval $[-2, -1]$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(-1) - f(-2)}{-1 - (-2)}$$ Substitute −1 for $x_2$ and −2 for $x_1$.

$$= \frac{[-(-1)^3 + 3(-1)] - [-(-2)^3 + 3(-2)]}{-1 - (-2)}$$ Evaluate $f(-1)$ and $f(-2)$

$$= \frac{-2 - 2}{-1 - (-2)} \text{ or } -4$$ Simplify.

The average rate of change on the interval $[-2, -1]$ is $-4$. Figure 13.4.1 supports this conclusion.

**b. [0, 1]**

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0}$$ Substitute 1 for $x_2$ and 0 for $x_1$.

$$= \frac{2 - 0}{1 - 0} \text{ or } 2$$ Evaluate $f(1)$ and $f(0)$ and simplify.

The average rate of change on the interval [0, 1] is 2. Figure 13.4.1 supports this conclusion.

### Guided Practice

**Find the average rate of change of each function on the given interval.**

**5A.** $f(x) = x^3 - 2x^2 - 3x + 2$; [2, 3] **6**

**5B.** $f(x) = x^4 - 6x^2 + 4x$; [−5, −3] **−220**

Average rate of change has many real-world applications. One common application involves the average speed of an object traveling over a distance $d$ or from a height $h$ in a given period of time $t$. Because speed is distance traveled per unit time, the average speed of an object cannot be negative.

## Real-World Example 6 Find Average Speed

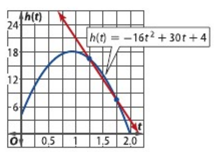
**PHYSICS** **The height of an object that is thrown straight up from a height of 4 ft above ground is given by $h(t) = -16t^2 + 30t + 4$, where $t$ is the time in seconds after the object is thrown. Find and interpret the average speed of the object from 1.25 to 1.75 seconds.**

$$\frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{h(1.75) - h(1.25)}{1.75 - 1.25}$$ Substitute 1.75 for $t_2$ and 1.25 for $t_1$.

$$= \frac{[-16(1.75)^2 + 30(1.75) + 4] - [-16(1.25)^2 + 30(1.25) + 4]}{0.5}$$ Evaluate $h(1.75)$ and $h(1.25)$.

$$= \frac{7.5 - 16.5}{0.5} \text{ or } -18$$ Simplify.

The average rate of change on the interval is −18. Therefore, the average *speed* of the object from 1.25 to 1.75 seconds is 18 ft/s, and the distance the object is from the ground is decreasing on average over that interval, as shown in the figure at the right.




### Real-World Link

Due to air resistance, a falling object will eventually reach a constant velocity known as *terminal velocity*. A skydiver with a closed parachute typically reaches terminal velocity of 190 to 240 km/h.

**Source:** *MSN Encarta*

### Guided Practice

**6.** **PHYSICS** If wind resistance is ignored, the distance $d(t)$ in feet an object travels when dropped from a high place is given by $d(t) = 16t^2$, where $t$ is the time is seconds after the object is dropped. Find and interpret the average speed of the object from 2 to 4 seconds. **96 ft/s; From 2 to 4 seconds, the distance that the object traveled increased on average over the interval.**





## Additional Examples

**5** Find the average rate of change of $f(x) = -2x^2 + 4x + 6$ on each interval.

**a.** [−3, −1] 12

**b.** [2, 5] −10

**6** **GRAVITY** The formula for the distance traveled by falling objects on the Moon is $d(t) = 0.83t^2$, where $d(t)$ is the distance in meters and $t$ is the time in seconds. Find and interpret the average speed of the object for each time interval.

**a.** 1 to 2 seconds 2.49 m/s

**b.** 2 to 3 seconds 4.15 m/s

## Follow-up

Students have explored characteristics of functions.

**Ask:**

- How does understanding parent functions and transformations help you to represent mathematical ideas and analyze real-world situations? Sample answer: Understanding the relationship between parent functions allows you to choose an appropriate function that could be used to represent a real-world situation.
- What characteristics of functions can help you analyze real-world situations? Explain. Sample answer: End behavior represents future behavior; critical points represent maximum and minimum values; average rates of change represent speeds and other changes.

## Differentiated Instruction AL OL

**Visual/Spatial Learners** Have students use the Internet to find photographs of the Teton Range in Grand Teton National Park. Each student should outline the skyline of the photograph that he or she selected. Ask students to identify the peaks and label them as either relative or absolute maxima.

# 3 Practice

## Formative Assessment

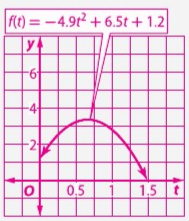
Use Exercises 1–47 to check for understanding.

Then use the table below to customize your assignments for students.

## Additional Answers

1. $f$ is increasing on $(-\infty, -0.5)$, decreasing on $(-0.5, 1)$, and increasing on $(1, \infty)$.
2. $f$ is decreasing on $(-\infty, -2.5)$, increasing on $(-2.5, 0)$, and decreasing on $(0, \infty)$.
3. $f$ is decreasing on $(-\infty, 2.5)$ and increasing on $(2.5, \infty)$.
4. $f$ is decreasing on $(-\infty, -1.5)$, increasing on $(-1.5, 1.5)$, and decreasing on $(1.5, \infty)$.
5. $f$ is increasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.
6. $f$ is increasing on $(-\infty, -6)$, decreasing on $(-6, -3)$, decreasing on $(-3, 0)$, and increasing on $(0, \infty)$.
7. $f$ is increasing on $(-\infty, -2)$, decreasing on $(0, 4)$, and increasing on $(4, \infty)$.
8. $f$ is increasing on $(-\infty, -4)$, decreasing on $(-4, 4)$, increasing on $(4, 5)$, and decreasing on $(5, \infty)$.
9. $f$ is constant on $(-\infty, -5)$, increasing on $(-5, -3.5)$, and decreasing on $(-3.5, \infty)$.
10. $f$ is increasing on $(-\infty, \infty)$.

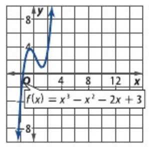
**11a.**



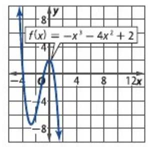

## Exercises

**Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Support the answer numerically.** (Example 1) **1–10. See margin.**

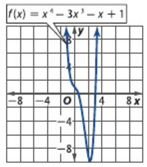
1.



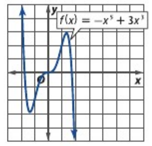

2.



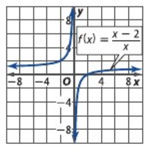

3.



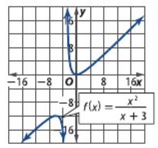

4.



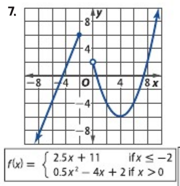

5.




6.




7.

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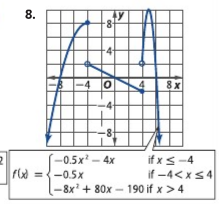


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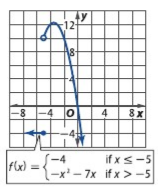


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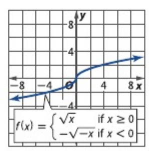
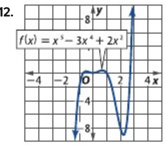



11. **BASKETBALL** The height of a free-throw attempt can be modeled by $f(t) = -4.9t^2 + 6.5t + 1.2$, where $t$ is time in seconds and $f(t)$ is the height in meters. (Example 2)
   a. Graph the height of the ball. **See margin.**
   b. Estimate the greatest height reached by the ball. Support the answer numerically. **about 4.23 m**

**12–21. See Chapter 11 Answer Appendix.**

**Estimate and classify the extrema for the graph of each function. Support the answers numerically.** (Example 2)

12.

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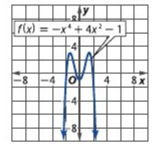


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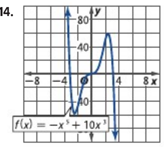


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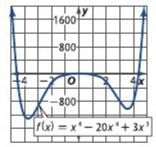


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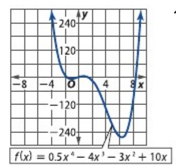


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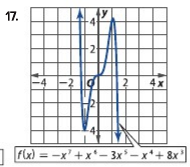


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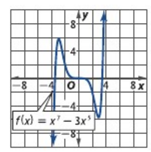


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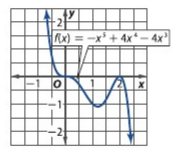


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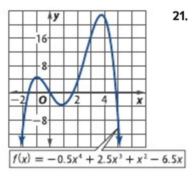


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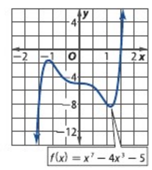







## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–47, 68–72, 74–93 | 1–47 odd, 94–97 | 2–46 even, 68–72, 74–93 |
| OL On Level | 1–47 odd, 48–53, 55–67 odd, 68–72, 74–97 | 1–47, 94–97 | 48–72, 74–93 |
| BL Beyond Level | 48–97 | | |


736 | Lesson 11-4 | Extrema and Average Rates of Change

**GRAPHING CALCULATOR** **Approximate to the nearest hundredth the relative or absolute extrema of each function. State the $x$-value(s) where they occur.** (Example 3)

22. $f(x) = 3x^3 - 6x^2 + 8$ **22–31. See margin.**

23. $g(x) = -2x^3 + 7x - 5$

24. $f(x) = -x^4 + 3x^3 - 2$

25. $f(x) = x^4 - 2x^2 + 5x$

26. $f(x) = x^5 - 2x^3 - 6x - 2$

27. $f(x) = -x^5 + 3x^2 + x - 1$

28. $g(x) = x^6 - 4x^4 + x$

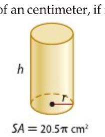
29. $g(x) = x^7 + 6x^2 - 4$

30. $f(x) = 0.008x^5 - 0.05x^4 - 0.2x^3 + 1.2x^2 - 0.7x$

31. $f(x) = 0.025x^5 - 0.1x^4 + 0.57x^3 + 1.2x^2 - 3.5x - 2$

32. **GRAPHIC DESIGN** A graphic designer wants to create a rectangular graphic that has a 2 cm margin on each side and a 4 cm margin on the top and the bottom. The design, including the margins, should have an area of 392 cm$^2$. What overall dimensions will maximize the size of the design, excluding the margins? (*Hint*: If one side of the design is $x$, then the other side is 392 divided by $x$.) (Example 4) **14 cm by 28 cm**

33. **GEOMETRY** Determine the radius and height that will maximize the volume of the cylinder shown. Round to the nearest hundredth of an centimeter, if necessary. (Example 4) **$r = 1.85$ cm; $h = 3.70$ cm**




**Find the average rate of change of each function on the given interval.** (Example 5)

34. $g(x) = -4x^2 + 3x - 4$; $[-1, 3]$ **$-5$**

35. $g(x) = 3x^2 - 8x + 2$; $[4, 8]$ **28**

36. $f(x) = 3x^3 - 2x^2 + 6$; $[2, 6]$ **140**

37. $f(x) = -2x^3 - 4x^2 + 2x - 8$; $[-2, 3]$ **$-16$**

38. $f(x) = 3x^4 - 2x^2 + 6x - 1$; $[5, 9]$ **4430**

39. $f(x) = -2x^4 - 5x^3 + 4x - 6$; $[-1, 5]$ **$-309$**

40. $h(x) = -x^5 - 5x^2 + 6x - 9$; $[3, 6]$ **$-2550$**

41. $h(x) = x^5 + 2x^4 + 3x - 12$; $[-5, -1]$ **472**

42. $f(x) = \frac{x-3}{x}$; $[5, 12]$ **0.05**

43. $f(x) = \frac{x+5}{x-4}$; $[-6, 2]$ **$-0.45$**

44. $f(x) = \sqrt{x+8}$; $[-4, 4]$ **$\approx 0.183$**

45. $f(x) = \sqrt{x-6}$; $[8, 16]$ **$\approx 0.219$**

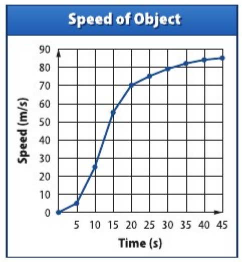


46. **WEATHER** The average high temperature by month in Dubai can be modeled by $f(x) = -0.5x^2 + 5x + 23$, where $x$ is the month and $x = 1$ represents January. Find the average rate of change for each time interval, and explain what this rate represents. (Example 6)
   a. April to May
   b. July to November
   **a–b. See margin.**

47. **COFFEE** Coffee consumption from 1990 to 2000 can be modeled by $f(x) = -0.004x^4 + 0.077x^3 - 0.38x^2 + 0.46x + 12$, where $x$ is the year, $x = 0$ corresponds with 1990, and the consumption is measured in millions of kilograms. Find the average rate of change for each time interval. (Example 6) **0.735 million kilograms more per year**
   a. 1990 to 2000
   b. 1995 to 2000
   **0.36 million kilograms more per year**

48. **TOURISM** Tourism in Abu Dhabi for a given year can be modeled using $f(x) = 0.0635x^6 - 2.49x^5 + 37.67x^4 - 275.3x^3 + 986.6x^2 - 1547.1x + 1390.5$, where $1 \le x \le 12$, $x$ represents the month, $x = 1$ corresponds with May 1st, and $f(x)$ represents the number of tourists in thousands.
   a. Graph the equation. **See margin.**
   b. During which month did the number of tourists reach its absolute maximum? **July**
   c. During which month did the number of tourists reach a relative maximum? **December**

49. Use the graph to complete the following.




   a. Find the average rate of change for [5, 15], [15, 20], and [25, 45]. **[5, 15]: 5; [15, 20]: 3; [25, 45]: 0.5**
   b. Compare and contrast the nature of the speed of the object over these time intervals. **See margin.**
   c. What conclusions can you make about the magnitude of the rate of change, the steepness of the graph, and the nature of the function? **See margin.**

50. **TECHNOLOGY** A computer company's research team determined that the profit per chip for a new processor chip can be modeled by $P(x) = -x^3 + 5x^2 + 8x$, where $x$ is the sales price of the chip in hundreds of dirhams.
   a. Graph the function. **See margin.**
   b. What is the optimum price per chip? **AED 400**
   c. What is the profit per chip at the optimum price? **AED 48**



## WatchOut!

**Common Error** In Exercises 32, 33, and 53, students may struggle to find a function with just one independent variable. Remind them that using systems of equations and substitution can reduce the number of independent variables.

## Additional Answers

**22.** rel. max: (0, 8); rel. min: (1.33, 4.44)

**23.** rel. max: (1.08, 0.04); rel. min: (−1.08, −10.04)

**24.** abs. max: (2.25, 6.54)

**25.** abs. min: (−1.38, −7.08)

**26.** rel. max: (−1.36, 6.54); rel. min: (1.36, −10.54)

**27.** rel. max: (1.11, 2.12); rel. min: (−0.17, −1.08)

**28.** rel. max: (0.41, 0.30); rel. min: (1.62, −7.85); abs. min: (−1.64, −11.12)

**29.** rel. max: (−1.11, 1.32); rel. min: (0, −4)

**30.** rel. max: (2.49, 1.45), (−3.72, 14.23); rel. min: (0.32, −0.11), (5.90, −6.83)

**31.** rel. max: (−1.66, 3.43); rel. min: (0.93, −3.82)

**46a.** 0.5° per month; The average temperature increases from early spring to mid-spring.

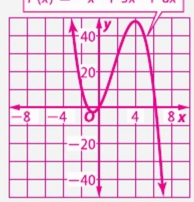
**46b.** −4° per month; The average temperature decreases from summer to late autumn.

**48a.** $f(x) = 0.0635x^6 - 2.49x^5 + 37.67x^4 - 275.3x^3 + 986.6x^2 - 1547.1x + 1390.5$

**49b.** The object is increasing in speed, or accelerating, along all three intervals. It is accelerating at the fastest rate for the interval [5, 15]. While it is very slowly accelerating for [25, 45], it is *still* increasing its speed.

**49c.** Steep graph = high magnitude rate of change = rapidly increasing or decreasing; flat graph = low magnitude rate of change = minimal increasing or decreasing.

**50a.** $P(x) = -x^3 + 5x^2 + 8x$



737

## Additional Answers

**76.** Continuous; $f(-3) = \sqrt{7}$ or about 2.65, $\lim_{x \to -3} f(x) \approx 2.65$, and $\lim_{x \to -3} f(x) = f(-3)$.

**77.** Continuous; $f(3) = 2$, $\lim_{x \to 3} f(x) = 2$, and $\lim_{x \to 3} f(x) = f(3)$.

**51.** **INCOME** Average net personal income for employees of a company from 1997 to 2007 can be modeled by $I(x) = -1.465x^5 + 35.51x^4 - 277.99x^3 + 741.06x^2 + 847.8x + 25362$, $0 \leq x \leq 10$, where $x$ is the number of years since 1997. **a–c. See Chapter 11 Answer Appendix.**

a. Graph the equation.

b. What was the average rate of change from 2000 to 2007? What does this value represent?

c. In what 4-year period was the average rate of change highest? lowest?

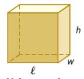
**52.** **BUSINESS** A company manufactures rectangular aquariums that have a capacity of 0.4 cubic meters. The glass used for the base of each aquarium is AED 4 per square meter. The glass used for the sides is AED 7 per square meter.

a. If the height and width of the aquarium are equal, find the dimensions that will minimize the cost to build an aquarium. **$w \approx 0.7$ m, $h \approx 0.7$ m, $\ell \approx 0.9$ m**

b. What is the minimum cost? **AED 163.96**

c. If the company also manufactures a cube-shaped aquarium with the same capacity, what is the difference in manufacturing costs? **AED 3.76**

**53.** **PACKAGING** Khalid needs to design an enclosed box with a square base and a volume of 3024 cubic centimeters. What dimensions minimize the surface area of the box? Support your reasoning. **See Chapter 11 Answer Appendix.**




**54–59. See Chapter 11 Answer Appendix.**

**Sketch a graph of a function with each set of characteristics.**

**54.** $f(x)$ is continuous and always increasing.

**55.** $f(x)$ is continuous and always decreasing.

**56.** $f(x)$ is continuous, always increasing, and $f(x) > 0$ for all values of $x$.

**57.** $f(x)$ is continuous, always decreasing, and $f(x) > 0$ for all values of $x$.

**58.** $f(x)$ is continuous, increasing for $x < -2$ and decreasing for $x > -2$.

**59.** $f(x)$ is continuous, decreasing for $x < 0$ and increasing for $x > 0$.

**Determine the coordinates of the absolute extrema of each function. State whether each extremum is a *maximum* or *minimum* value.**

**60.** $f(x) = 2(x - 3)^2 + 5$ **(3, 5); minimum**

**61.** $f(x) = -0.5(x + 5)^2 - 1$ **(−5, −1); maximum**

**62.** $f(x) = -4|x - 22| + 65$ **(22, 65); maximum**

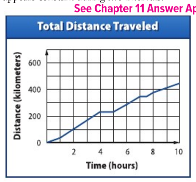
**63.** $f(x) = 4(3x - 7)^4 + 8$ **(2.3, 8); minimum**

**64.** $f(x) = (36 - x^2)^{0.5}$ **(0, 6); maximum**

**65.** $f(x) = -(25 - x^2)^{0.5}$ **(0, −5); minimum**

**66.** $f(x) = x^3 + x$ **no extrema**

**67.** **TRAVEL** Each hour, Saeed recorded and graphed the total distance in kilometers his family drove during a trip. Give some reasons as to why the average rate of change varies and even appears constant during two intervals. **See Chapter 11 Answer Appendix.**




**68.** **POINTS OF INFLECTION** Determine which of the graphs in Exercises 1–10 and 12–21 have points of inflection that are critical points, and estimate the location of these points on each graph.

**Exercise 3: (0, 1), Exercise 4: (0, 0), Exercise 10: (0, 0), Exercise 14: (0, 0), Exercise 17: (0, 0), Exercise 18: (0, 0), Exercise 19: (0, 0), Exercise 21: (0, −5)**

### H.O.T. Problems — Use Higher-Order Thinking Skills

**OPEN ENDED** **Sketch a graph of a function with each set of characteristics.** **69–70. See Chapter 11 Answer Appendix.**

**69.** infinite discontinuity at $x = -2$
increasing on $(-\infty, -2)$
increasing on $(-2, \infty)$
$f(-6) = -6$

**70.** continuous
average rate of change for $[3, 8]$ is 4
decreasing on $(8, \infty)$
$f(-4) = 2$

**71.** **REASONING** What is the slope of the secant line from $(a, f(a))$ to $(b, f(b))$ when $f(x)$ is constant for the interval $[a, b]$? Explain your reasoning. **See Chapter 11 Answer Appendix.**

**72.** **REASONING** If the average rate of change of $f(x)$ on the interval $(a, b)$ is positive, is $f(x)$ *sometimes*, *always*, or *never* increasing on $(a, b)$? Explain your reasoning. **See Chapter 11 Answer Appendix.**

**73.** **CHALLENGE** Use a calculator to graph $f(x) = \sin x$ in degree mode. Describe the relative extrema of the function and the window used for your graph. **See Chapter 11 Answer Appendix.**

**74.** **REASONING** A continuous function $f$ has a relative minimum at $c$ and is increasing as $x$ increases from $c$. Describe the behavior of the function as $x$ increases to $c$. Explain your reasoning. **See Chapter 11 Answer Appendix.**

**75.** **WRITING IN MATH** Describe how the average rate of change of a function relates to a function when it is increasing, decreasing, and constant on an interval. **See Chapter 11 Answer Appendix.**





738 | Lesson 11-4 | Extrema and Average Rates of Change

## Spiral Review

**Determine whether each function is continuous at the given $x$-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.** (Lesson 11-3) **76–78. See margin.**

**76.** $f(x) = \sqrt{x^2 - 2}$; $x = -3$

**77.** $f(x) = \sqrt{x + 1}$; $x = 3$

**78.** $h(x) = \frac{x^2 - 25}{x + 5}$; $x = -5$ and $x = 5$

**GRAPHING CALCULATOR** **Graph each function. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm algebraically. If odd or even, describe the symmetry of the graph of the function.** (Lesson 11-2) **79–81. See margin.**

**79.** $f(x) = |x^5|$

**80.** $f(x) = \frac{x + 8}{x - 4}$

**81.** $g(x) = \frac{x^2}{x + 3}$

**State the domain of each function.** (Lesson 11-1)

**82.** $f(x) = \frac{3x}{x^2 - 5}$ $\{x \mid x \neq \pm\sqrt{5}, x \in \mathbb{R}\}$

**83.** $g(x) = \sqrt{x^2 - 9}$ $(-\infty, -3] \cup [3, \infty)$

**84.** $h(x) = \frac{x + 2}{\sqrt{x^2 - 7}}$ $(-\infty, -\sqrt{7}) \cup (\sqrt{7}, \infty)$

**85.** Find the values of $x$, $y$, and $z$ for $3\begin{bmatrix} x & y - 1 \\ 4 & 3z \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 6z & 3x + y \end{bmatrix}$. **(5, 3, 2)**

**86.** If possible, find the solution of $y = x + 2z$, $z = -1 - 2x$, and $x = y - 14$. **(−4, 10, 7)**

**Solve each equation.**

**87.** $x^2 + 3x - 18 = 0$ **−6, 3**

**88.** $2a^2 + 11a - 21 = 0$ $-7, \frac{3}{2}$

**89.** $z^2 - 4z - 21 = 0$ **7, −3**

**Simplify.**

**90.** $i^{19}$ $-i$

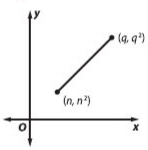
**91.** $(7 - 4i) + (2 - 3i)$ $9 - 7i$

**92.** $\left(\frac{1}{2} + i\right) - (2 - i)$ $-\frac{3}{2} + 2i$

**93.** **ELECTRICITY** The number of volts $E$ produced by a circuit is given by $E = I \cdot Z$, where $I$ is the current in amps and $Z$ is the impedance in ohms. What number of amps is needed in a circuit that has an impedance of $3 - j$ ohms in order to produce $21 + 12j$ volts? **$5 + 6j$ amps**

## Skills Review for Standardized Tests

**94.** **SAT/ACT** In the figure, if $q \neq n$, what is the slope of the line segment? **A**




A $q + n$

B $q - n$

C $\frac{q^2 + q}{n^2 - n}$

D $\frac{1}{q + n}$

E $\frac{1}{q - n}$

**95.** **REVIEW** When the number of a year is divisible by 4, then a leap year occurs. However, when the year is divisible by 100, then a leap year does not occur unless the year is divisible by 400. Which is an example of a leap year? **H**

F 1882

G 1900

H 2000

J 2100

**96.** The function $f(x) = x^3 + 2x^2 - 4x - 6$ has a relative maximum and relative minimum located at which of the following $x$-values? **C**

A relative maximum at $x \approx -0.7$, relative minimum at $x \approx 2$

B relative maximum at $x \approx -0.7$, relative minimum at $x \approx -2$

C relative maximum at $x \approx -2$, relative minimum at $x \approx 0.7$

D relative maximum at $x \approx 2$, relative minimum at $x \approx 0.7$

**97.** **REVIEW** A window is in the shape of an equilateral triangle. Each side of the triangle is 2.5 m long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support? **G**

F 1.74 m

G 2.2 m

H 3.4 m

J 4.2 m





## Differentiated Instruction BL

**Extension** A line that is tangent to the graph of a polynomial function at a maximum or minimum will be horizontal. Suppose two points defining a secant line for the graph of a polynomial function converge on a relative maximum. What happens to the slope of the secant line as the points get very close to the maximum? How is this related to the tangent line at the relative maximum? The slope of the secant line approaches 0. As the points get closer to the relative maximum, the secant line gets closer to being a horizontal tangent.

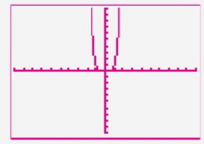
## 4 Assess

**Crystal Ball** Ask students to describe how they think today's lesson will connect with the next lesson on parent functions and transformations.

### Additional Answers

**78.** Discontinuous at $x = -5$; $h(-5)$ is undefined and $\lim_{x \to -5} h(x) = -10$, so $h(x)$ has a removable discontinuity at $x = -5$. Continuous at $x = 5$; $h(5) = 0$, $\lim_{x \to 5} h(x) = 0$, and $\lim_{x \to 5} h(x) = h(5)$.

**79.**



[−10, 10] scl: 1 by [−10, 10] scl: 1

Even; the graph of $f(x)$ is symmetric with respect to the $y$-axis.

$f(-x) = |(-x)^5|$
$= |x^5| = f(x)$
$-f(x) = -|x^5| \neq f(-x)$

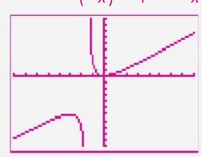
**80.**

[−4, 16] scl: 1 by [−13, 17] scl: 3

neither;

$f(-x) = \frac{(-x) + 8}{(-x) - 4}$ or $\frac{8 - x}{-x - 4}$

**81.**



[−16, 16] scl: 2 by [−22, 18] scl: 2

neither;

$g(-x) = \frac{(-x)^2}{-x + 3}$ or $\frac{x^2}{-x + 3}$

## Lessons 11-1 to 11-4

### Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half of the chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

### Additional Answers

**6b.** $[0, 3.22]$; Sample answer: The relevant domain represents the interval of time beginning when the ball was hit and ending when it reached the ground. Because time cannot be negative and the height of the ball is 0 when $t = 3.22$, $0 \le t \le 3.22$.

**7.** $y$-intercept: 0; zeros: $-4, 0, 4$;

$$x^3 - 16x = 0$$
$$x(x^2 - 16) = 0$$
$$x(x + 4)(x - 4) = 0$$
$$x = 0 \text{ or } x + 4 = 0 \text{ or } x - 4 = 0$$
$$x = -4 \qquad x = 4$$

**8.** $y$-intercept: 5; zero: 25;

$$5 - \sqrt{x} = 0$$
$$5 = \sqrt{x}$$
$$25 = x$$

**9.** $D = [0, \infty)$, $R = [0, \infty)$

**10.** $D = \{x \mid x \in \mathbb{R}\}$, $R = \{y \mid y \in \mathbb{Z}\}$

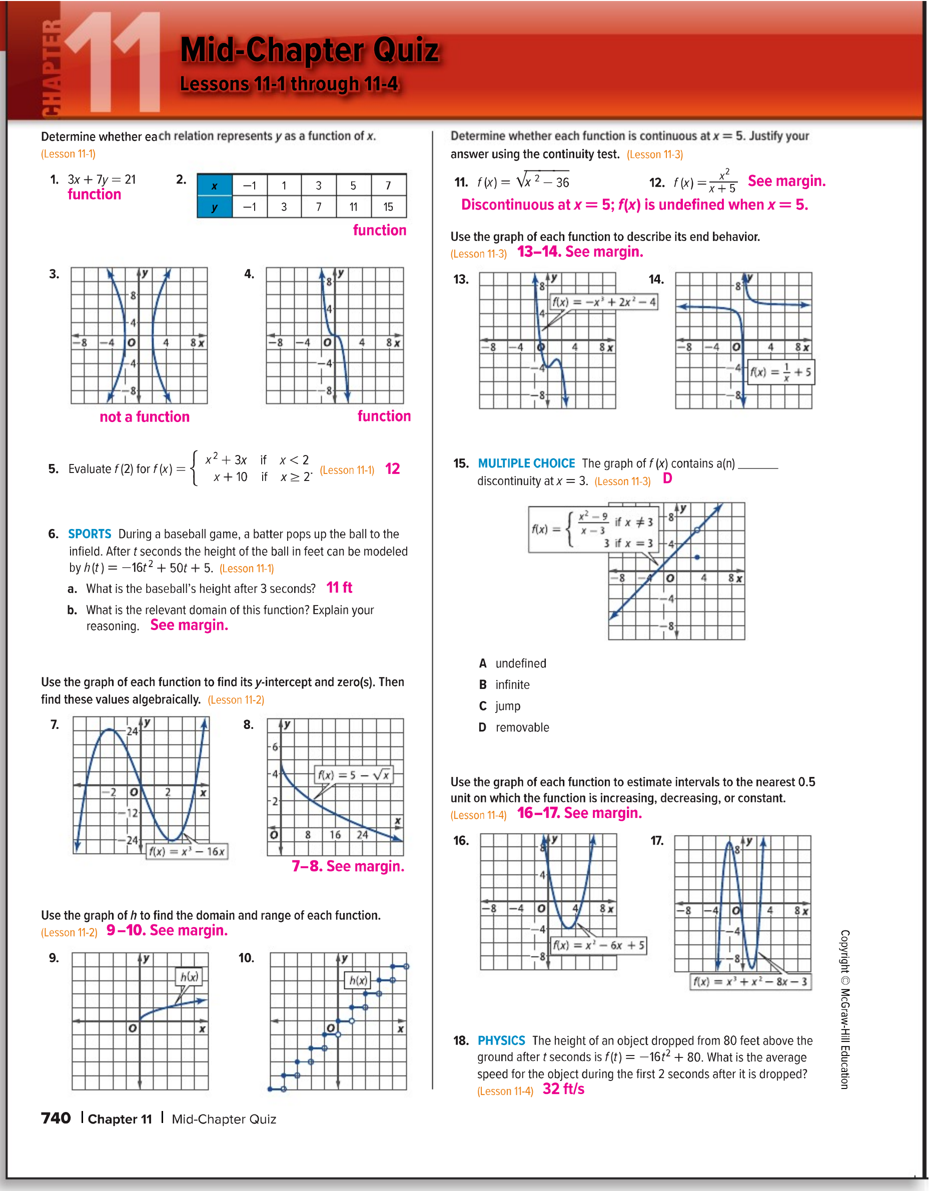
**12.** Continuous at $x = 5$; $f(5) = 2.5$, $\lim_{x \to 5} f(x) = 2.5$, and $\lim_{x \to 5} f(x) = f(5)$.

**13.** From the graph, it appears that $f(x) \to -\infty$ as $x \to \infty$, and $f(x) \to \infty$ as $x \to -\infty$.

**14.** From the graph, it appears that $f(x) \to 5$ as $x \to \infty$, and $f(x) \to 5$ as $x \to -\infty$.

**16.** $f$ is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

**17.** $f$ is increasing on $(-\infty, -2)$, decreasing on $(-2, 1.5)$, and increasing on $(1.5, \infty)$.



CHAPTER 11

## Mid-Chapter Quiz

Lessons 11-1 through 11-4

Determine whether each relation represents $y$ as a function of $x$. (Lesson 11-1)

1. $3x + 7y = 21$ **function**

2.

| $x$ | $-1$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|---|
| $y$ | $-1$ | 3 | 7 | 11 | 15 |

**function**

3. **not a function**

4. **function**

5. Evaluate $f(2)$ for $f(x) = \begin{cases} x^2 + 3x & \text{if } x < 2 \\ x + 10 & \text{if } x \ge 2 \end{cases}$. (Lesson 11-1) **12**

6. **SPORTS** During a baseball game, a batter pops up the ball to the infield. After $t$ seconds the height of the ball in feet can be modeled by $h(t) = -16t^2 + 50t + 5$. (Lesson 11-1)

   a. What is the baseball's height after 3 seconds? **11 ft**

   b. What is the relevant domain of this function? Explain your reasoning. **See margin.**

Use the graph of each function to find its $y$-intercept and zero(s). Then find these values algebraically. (Lesson 11-2)

7. $f(x) = x^3 - 16x$

8. $f(x) = 5 - \sqrt{x}$

**7–8. See margin.**

Use the graph of $h$ to find the domain and range of each function. (Lesson 11-2) **9–10. See margin.**

9. $h(x)$

10. $h(x)$

Determine whether each function is continuous at $x = 5$. Justify your answer using the continuity test. (Lesson 11-3)

11. $f(x) = \sqrt{x^2 - 36}$ **Discontinuous at $x = 5$; $f(x)$ is undefined when $x = 5$.**

12. $f(x) = \frac{x^2}{x + 5}$ **See margin.**

Use the graph of each function to describe its end behavior. (Lesson 11-3) **13–14. See margin.**

13. $f(x) = -x^3 + 2x^2 - 4$

14. $f(x) = \frac{1}{x} + 5$

15. **MULTIPLE CHOICE** The graph of $f(x)$ contains a(n) ______ discontinuity at $x = 3$. (Lesson 11-3) **D**

$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \ne 3 \\ 3 & \text{if } x = 3 \end{cases}$

A undefined

B infinite

C jump

D removable

Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. (Lesson 11-4) **16–17. See margin.**

16. $f(x) = x^2 - 6x + 5$

17. $f(x) = x^3 + x^2 - 8x - 3$

18. **PHYSICS** The height of an object dropped from 80 feet above the ground after $t$ seconds is $f(t) = -16t^2 + 80$. What is the average speed for the object during the first 2 seconds after it is dropped? (Lesson 11-4) **32 ft/s**

740 | Chapter 11 | Mid-Chapter Quiz

# LESSON 11-5 Parent Functions and Transformations

**Then**

- You analyzed graphs of functions. (Lessons 11-2 through 11-4)

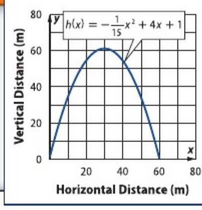
**Now**

1. Identify, graph, and describe parent functions.
2. Identify and graph transformations of parent functions.

**Why?**

- The path of a 60 m free kick can be modeled by the function at the right. This function is related to the basic quadratic function $f(x) = x^2$.

**Free kick**




**NewVocabulary**

parent function
constant function
zero function
identity function
quadratic function
cubic function
square root function
reciprocal function
absolute value function
step function
greatest integer function
transformation
translation
reflection
dilation

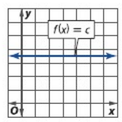
## 1 Parent Functions

A *family of functions* is a group of functions with graphs that display one or more similar characteristics. A **parent function** is the simplest of the functions in a family. This is the function that is transformed to create other members in a family of functions.

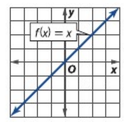
In this lesson, you will study eight of the most commonly used parent functions. You should already be familiar with the graphs of the following linear and polynomial parent functions.

### KeyConcept Linear and Polynomial Parent Functions

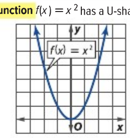
A **constant function** has the form $f(x) = c$, where $c$ is any real number. Its graph is a horizontal line. When $c = 0$, $f(x)$ is the **zero function**.



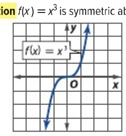

The **identity function** $f(x) = x$ passes through all points with coordinates $(a, a)$.




The **quadratic function** $f(x) = x^2$ has a U-shaped graph.



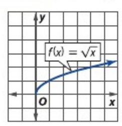

The **cubic function** $f(x) = x^3$ is symmetric about the origin.



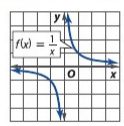

You should also be familiar with the graphs of both the square root and reciprocal functions.

### KeyConcept Square Root and Reciprocal Parent Functions

The **square root function** has the form $f(x) = \sqrt{x}$.




The **reciprocal function** has the form $f(x) = \frac{1}{x}$.








## 1 Focus

**VerticalAlignment**

**Before Lesson 11-5** Analyze graphs of functions.

**Lesson 11-5** Identify, graph, and describe parent functions.
Identify and graph transformations of parent functions.

**After Lesson 11-5** Perform operations with and find compositions of functions.

## 2 Teach

### Scaffolding Questions

Have students read the **Why?** section of the lesson.

**Ask:**

- What are the similarities and differences between $f(x) = x$ and $g(x) = x + 2$? The slope of the lines is the same, with $g(x)$ shifted up 2 units.
- Describe how different values of $a$ will affect the graph of $f(x) = x + a$. The value of $a$ shifts the line up or down $|a|$ units.

*(continued on the next page)*

- What are the similarities and differences between $f(x) = x^2$ and $g(x) = x^2 + 2$? The shape of the parabolas is the same, with $g(x)$ shifted up 2 units.

## 1 Parent Functions

**Example 1** shows how to describe important characteristics of a function.

### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

**Additional Example**

**1** Describe the following characteristics of the graph of the parent function $f(x) = \frac{1}{x}$: domain, range, intercepts, symmetry, continuity, end behavior, and intervals on which the graph is increasing/decreasing. D and R: $(-\infty, 0) \cup (0, \infty)$; no intercepts. The graph is symmetric about the origin, so it is odd. The graph is continuous for all values in its domain with an infinite discontinuity at $x = 0$. The graph decreases on both intervals of its domain.

Another parent function is the piecewise-defined absolute value function.

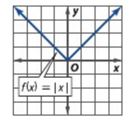
**KeyConcept** Absolute Value Parent Function

Words: The **absolute value function**, denoted $f(x) = |x|$, is a V-shaped function defined as

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Examples: $|-5| = 5$, $|0| = 0$, $|4| = 4$

Model




A piecewise-defined function in which the graph resembles a set of stairs is called a **step function**. The most well-known step function is the greatest integer function.

**StudyTip**

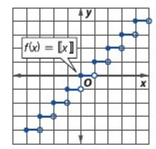
**Floor Function** The greatest integer function is also known as the *floor function*.

**KeyConcept** Greatest Integer Parent Function

Words: The **greatest integer function**, denoted $f(x) = [\![x]\!]$, is defined as the greatest integer less than or equal to $x$.

Examples: $[\![-4]\!] = -4$, $[\![-1.5]\!] = -2$, $[\![\frac{1}{3}]\!] = 0$

Model




**1.** $D = \{x \mid x \in \mathbb{R}\}$, $R = \{y \mid 0 \leq y, y \in \mathbb{R}\}$; the graph has one intercept at (0, 0), is symmetric with respect to the *y*-axis and therefore $f(x)$ is even, is continuous for all values in its domain, is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to \infty} f(x) = \infty$

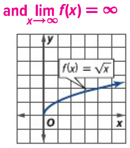



Figure 11.5.1

Using the tools you learned in Lessons 11-1 through 11-4, you can describe characteristics of each parent function. Knowing the characteristics of a parent function can help you analyze the shapes of more complicated graphs in that family.

**Example 1** Describe Characteristics of a Parent Function

**Describe the following characteristics of the graph of the parent function $f(x) = \sqrt{x}$: domain, range, intercepts, symmetry, continuity, end behavior, and intervals on which the graph is increasing/decreasing.**

The graph of the square root function (Figure 11.5.1) has the following characteristics.

- The domain of the function is $[0, \infty)$, and the range is $[0, \infty)$.
- The graph has one intercept at (0, 0).
- The graph has no symmetry. Therefore, $f(x)$ is neither odd nor even.
- The graph is continuous for all values in its domain.
- The graph begins at $x = 0$ and $\lim_{x \to \infty} f(x) = \infty$.
- The graph is increasing on the interval $(0, \infty)$.

**GuidedPractice**

1. Describe the following characteristics of the graph of the parent function $f(x) = |x|$: domain, range, intercepts, symmetry, continuity, end behavior, and intervals on which the graph is increasing/decreasing.

## 2 Transformations

**Transformations** of a parent function can affect the appearance of the parent graph. *Rigid transformations* change only the position of the graph, leaving the size and shape unchanged. *Nonrigid transformations* distort the shape of the graph.





### Focus on Mathematical Content

**Floor and Ceiling Functions** The greatest integer function or floor function can also be denoted $f(x) = \lfloor x \rfloor$. A similar function known as the *ceiling function*, denoted $f(x) = \lceil x \rceil$, is defined as the least integer greater than or equal to $x$. For example, $\lceil 2 \rceil = 2$, $\lceil 2.1 \rceil = 3$, and $\lceil 2.9 \rceil = 3$.

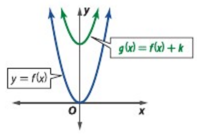
A **translation** is a rigid transformation that has the effect of shifting the graph of a function. A *vertical translation* of a function $f$ shifts the graph of $f$ up or down, while a *horizontal translation* shifts the graph left or right. Horizontal and vertical translations are examples of rigid transformations.

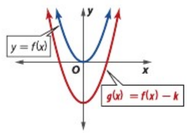
**KeyConcept** Vertical and Horizontal Translations

**Vertical Translations**

The graph of $g(x) = f(x) + k$ is the graph of $f(x)$ translated

- $k$ **units up** when $k > 0$, and
- $k$ **units down** when $k < 0$.

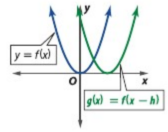


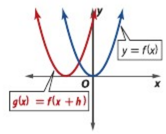




**Horizontal Translations**

The graph of $g(x) = f(x - h)$ is the graph of $f(x)$ translated

- $h$ **units right** when $h > 0$, and
- $h$ **units left** when $h < 0$.




**Example 2** Graph Translations

**Use the graph of $f(x) = |x|$ to graph each function.**

**a.** $g(x) = |x| + 4$

This function is of the form $g(x) = f(x) + 4$. So, the graph of $g(x)$ is the graph of $f(x) = |x|$ translated 4 units up, as shown in Figure 11.5.2.

**b.** $g(x) = |x + 3|$

This function is of the form $g(x) = f(x + 3)$ or $g(x) = f\,[x - (-3)]$. So, the graph of $g(x)$ is the graph of $f(x) = |x|$ translated 3 units left, as shown in Figure 11.5.3.

**c.** $g(x) = |x - 2| - 1$

This function is of the form $g(x) = f(x - 2) - 1$. So, the graph of $g(x)$ is the graph of $f(x) = |x|$ translated 2 units right and 1 unit down, as shown in Figure 11.5.4.

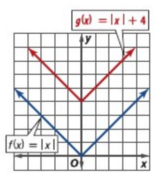



Figure 11.5.2

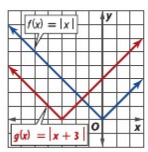



Figure 11.5.3

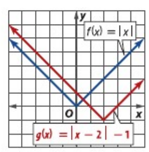



Figure 11.5.4

**GuidedPractice** Use the graph of $f(x) = x^3$ to graph each function. **2A–C. See margin.**

**2A.** $h(x) = x^3 - 5$ **2B.** $h(x) = (x - 3)^3$ **2C.** $h(x) = (x + 2)^3 + 4$

**TechnologyTip**

**Translations** You can translate a graph using a graphing calculator. Under [Y=], place an equation in Y1. Move to the Y2 line, and then press [VARS] [▶] [ENTER] [ENTER]. This will place Y1 in the Y2 line. Enter a number to translate the function. Press [GRAPH]. The two equations will be graphed in the same window.





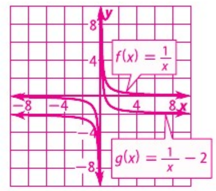
## 2 Transformations

**Example 2** shows how to graph translations. **Example 3** shows how to write equations for transformations. **Example 4** shows how to describe and graph transformations. **Example 5** shows how to graph a piecewise-defined function. **Examples 6 and 7** show how to use, describe, and graph transformations of equations.

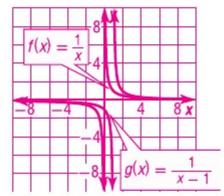
**Additional Example**

**2** Use the graph of $f(x) = \frac{1}{x}$ to graph each function.

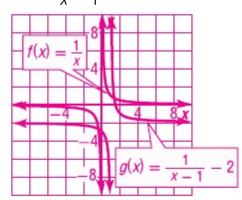
**a.** $g(x) = \frac{1}{x} - 2$




**b.** $g(x) = \frac{1}{x - 1}$

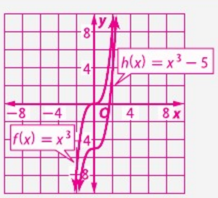



**c.** $g(x) = \frac{1}{x - 1} - 2$

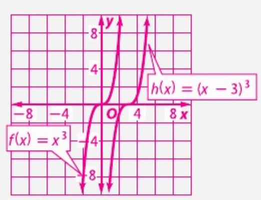



### Additional Answers (Guided Practice)

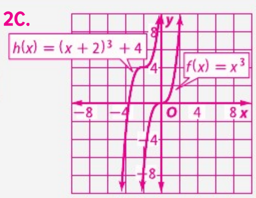
**2A.**




**2B.**




**2C.**

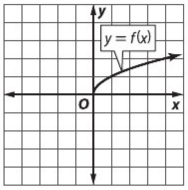



### Teach with Tech

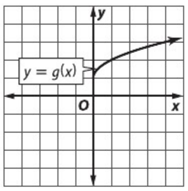
**Graphing Calculator** Have students use graphing calculators to adjust parameters in each parent function and note the effects of changing each parameter. Then have students work with a partner to compare their results.

## Additional Example

**3** Describe how the graphs of $f(x) = \sqrt{x}$ and $g(x)$ are related. Then write an equation for $g(x)$.

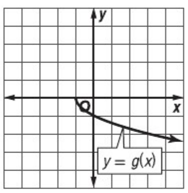



**a.**




The graph of $g(x)$ is the graph of $f(x)$ translated 1 unit up; $g(x) = \sqrt{x} + 1$.

**b.**




The graph of $g(x)$ is the graph of $f(x)$ translated 1 unit down and reflected in the $x$-axis; $g(x) = -\sqrt{x+1}$.

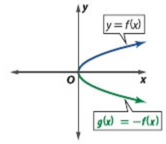
## Focus on Mathematical Content

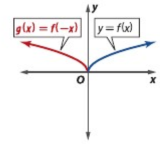
**Transforming Functions** When the graph of a function is translated or reflected, it keeps the same shape. When a graph is dilated, it changes shape. From a geometric view, translations and reflections of graphs preserve shape, so the image is congruent to the graph of the parent function. However, when a graph is dilated, it does not preserve the same curves. So, the graph of the function is not similar to the graph of the parent function.

Another type of rigid transformation is a **reflection**, which produces a mirror image of the graph of a function with respect to a specific line.

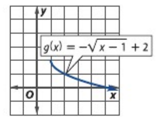
### KeyConcept Reflections in the Coordinate Axes

| Reflection in $x$-axis | Reflection in $y$-axis |
|---|---|
| $g(x) = -f(x)$ is the graph of $f(x)$ reflected in the $x$-axis. | $g(x) = f(-x)$ is the graph of $f(x)$ reflected in the $y$-axis. |

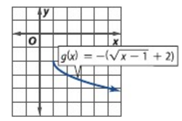



When writing an equation for a transformed function, be careful to indicate the transformations correctly. The graph of $g(x) = -\sqrt{x-1} + 2$ is different from the graph of $g(x) = -(\sqrt{x-1} + 2)$.




reflection of $f(x) = \sqrt{x}$ in the $x$-axis, then translated 1 unit to the right and 2 units up




translation of $f(x) = \sqrt{x}$ 1 unit to the right and 2 units up, then reflected in the $x$-axis

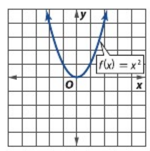



Figure 11.5.5

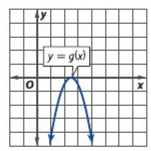
**3A. The graph of $g(x)$ is the graph of $f(x)$ reflected in the $x$-axis and translated 2 units down; $g(x) = -\frac{1}{x} - 2$.**

**3B. The graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the left and reflected in the $x$-axis; $g(x) = -\frac{1}{x+4}$.**

### Example 3 Write Equations for Transformations

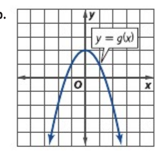
**Describe how the graphs of $f(x) = x^2$ and $g(x)$ are related. Then write an equation for $g(x)$.**

**a.**




The graph of $g(x)$ is the graph of $f(x) = x^2$ translated 2.5 units to the right and reflected in the $x$-axis. So, $g(x) = -(x - 2.5)^2$.

**b.**

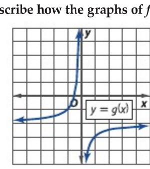



The graph of $g(x)$ is the graph of $f(x) = x^2$ reflected in the $x$-axis and translated 2 units up. So, $g(x) = -x^2 + 2$.

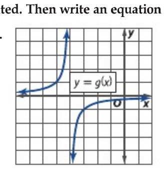
### Guided Practice

**Describe how the graphs of $f(x) = \frac{1}{x}$ and $g(x)$ are related. Then write an equation for $g(x)$.**

**3A.**




**3B.**








## DifferentiatedInstruction AL OL

**Visual Learners** Ask students to create posters displaying the eight parent functions studied in this lesson and how to transform them.

A **dilation** is a nonrigid transformation that has the effect of compressing (shrinking) or expanding (enlarging) the graph of a function vertically or horizontally.

**StudyTip**

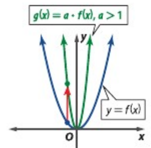
**Dilations** Sometimes pairs of dilations look similar such as a vertical expansion and a horizontal compression. It is not possible to tell which dilation a transformation is from the graph. You must compare the equation of the transformed function to the parent function.

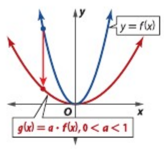
## KeyConcept Vertical and Horizontal Dilations

**Vertical Dilations**

If $a$ is a positive real number, then $g(x) = a \cdot f(x)$, is

- the graph of $f(x)$ **expanded vertically**, if $a > 1$.
- the graph of $f(x)$ **compressed vertically**, if $0 < a < 1$.

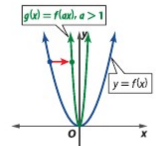


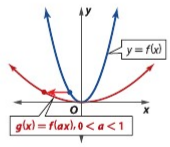




**Horizontal Dilations**

If $a$ is a positive real number, then $g(x) = f(ax)$, is

- the graph of $f(x)$ **compressed horizontally**, if $a > 1$.
- the graph of $f(x)$ **expanded horizontally**, if $0 < a < 1$.



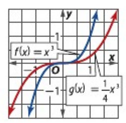




### Example 4 Describe and Graph Transformations

**Identify the parent function $f(x)$ of $g(x)$, and describe how the graphs of $g(x)$ and $f(x)$ are related. Then graph $f(x)$ and $g(x)$ on the same axes.**

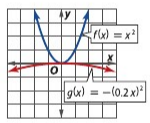
**a.** $g(x) = \frac{1}{4}x^3$

The graph of $g(x)$ is the graph of $f(x) = x^3$ compressed vertically because $g(x) = \frac{1}{4}x^3 = \frac{1}{4}f(x)$ and $0 < \frac{1}{4} < 1$.




**b.** $g(x) = -(0.2x)^2$

The graph of $g(x)$ is the graph of $f(x) = x^2$ expanded horizontally and then reflected in the $x$-axis because $g(x) = -(0.2x)^2 = -f(0.2x)$ and $0 < 0.2 < 1$.




**GuidedPractice** **4A–B. See margin.**

**4A.** $g(x) = [\![x]\!] - 4$

**4B.** $g(x) = \frac{15}{x} + 3$

You can use what you have learned about transformations of functions to graph a piecewise-defined function.

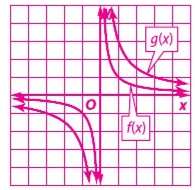




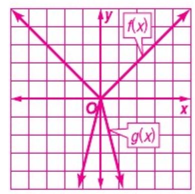
## Additional Example

**4** Identify the parent function $f(x)$ of $g(x)$, and describe how the graphs of $g(x)$ and $f(x)$ are related. Then graph $f(x)$ and $g(x)$ on the same axes.

**a.** $g(x) = \frac{3}{x}$ $f(x) = \frac{1}{x}$; The graph of $g(x)$ is the graph of $f(x)$ expanded vertically by a factor of 3.

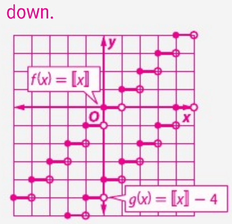



**b.** $g(x) = -|4x|$ $f(x) = |x|$; The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally by a factor of 4 and reflected in the $x$-axis.

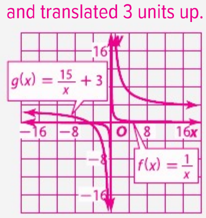



### Additional Answers (Guided Practice)

**4A.** $f(x) = [\![x]\!]$; The graph of $g(x)$ is the graph of $f(x)$ translated 4 units down.



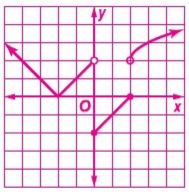

**4B.** $f(x) = \frac{1}{x}$; The graph of $g(x)$ is the graph of $f(x)$ expanded vertically by a factor of 15 and translated 3 units up.



## Additional Examples

**5** Graph

$$f(x) = \begin{cases} |x+2| & \text{if } x < 0 \\ |x| - 2 & \text{if } 0 \le x \le 2 \\ \sqrt{x-2} + 2 & \text{if } x > 2 \end{cases}.$$

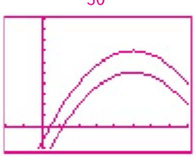



**6** **AMUSEMENT PARK** The "Wild Ride" roller coaster has a section that is shaped like the function $g(x) = \frac{-x^2}{30} + \frac{10x}{3} - \frac{100}{3}$, where $g(x)$ is the vertical distance in meters the roller coaster track is from the ground and $x$ is the horizontal distance in meters from the start of the ride.

**a.** Describe the transformations of the parent function $f(x) = x^2$ used to graph $g(x)$. $g(x)$ is the graph of $f(x)$ translated 50 units right, compressed vertically, reflected in the $x$-axis, and then translated 50 units up.

**b.** Suppose the ride designers decide to increase the highest point of the ride to 70 m. Rewrite $g(x)$ to reflect this change. Graph both functions on the same coordinate axes using a graphing calculator.

$g(x) = -\frac{1}{30}(x - 50)^2 + 70$



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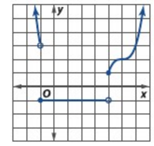
### Example 5 Graph a Piecewise-Defined Function

Graph $f(x) = \begin{cases} 3x^2 & \text{if } x < -1 \\ -1 & \text{if } -1 \le x < 4 \\ (x-5)^3 + 2 & \text{if } x \ge 4 \end{cases}$.

On the interval $(-\infty, -1)$, graph $y = 3x^2$.
On the interval $[-1, 4)$, graph the constant function $y = -1$.
On the interval $[4, \infty)$, graph $y = (x-5)^3 + 2$.

Draw circles at $(-1, 3)$ and $(4, -1)$ and dots at $(-1, -1)$ and $(4, 1)$ because $f(-1) = -1$ and $f(4) = 1$.




**Guided Practice**

Graph each function. 5A–B. See margin.

**5A.** $g(x) = \begin{cases} x - 5 & \text{if } x \le 0 \\ x^3 & \text{if } 0 < x \le 2 \\ \frac{2}{x} & \text{if } x > 2 \end{cases}$

**5B.** $h(x) = \begin{cases} (x+6)^2 & \text{if } x < -5 \\ 7 & \text{if } -5 \le x \le 2 \\ |4 - x| & \text{if } x > 2 \end{cases}$

You can also use what you have learned about transformations to transform functions that model real-world data or phenomena.

### Real-World Example 6 Transformations of Functions

**FOOTBALL** **The path of a 60 m free kick can be modeled by $g(x) = -\frac{1}{15}x^2 + 4x + 1$, where $g(x)$ is the vertical distance in meters of the football from the ground and $x$ is the horizontal distance in meters such that $x = 0$ corresponds to 20 m from the free kick taker's goal line.**

**a. Describe the transformations of the parent function $f(x) = x^2$ used to graph $g(x)$.**

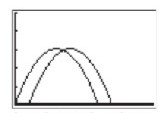
Rewrite the function so that it is in the form $g(x) = a(x - h)^2 + k$ by completing the square.

| | |
|---|---|
| $g(x) = -\frac{1}{15}x^2 + 4x + 1$ | Original function |
| $= -\frac{1}{15}(x^2 - 60x) + 1$ | Factor $-\frac{1}{15}x^2 + 4x$. |
| $= -\frac{1}{15}(x^2 - 60x + 900) + 1 + \frac{1}{15}(900)$ | Complete the square. |
| $= -\frac{1}{15}(x - 30)^2 + 61$ | Write $x^2 - 60x + 900$ as a perfect square and simplify. |

So, $g(x)$ is the graph of $f(x)$ translated 30 units right, compressed vertically, reflected in the $x$-axis, and then translated 61 units up.

**b. Suppose the free kick was taken from 30 m from the free kick taker's goal line. Rewrite $g(x)$ to reflect this change. Graph both functions on the same graphing calculator screen.**

A change of position from the 20 to 30 m from the free kick taker's goal line is a horizontal translation of 10 m to the right, so subtract an additional 10 m from inside the squared expression.



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$g(x) = -\frac{1}{15}(x - 30 - 10)^2 + 61$ or $g(x) = -\frac{1}{15}(x - 40)^2 + 61$

**Guided Practice**

**6. ELECTRICITY** The current in amps flowing through a DVD player is described by $I(x) = \sqrt{\frac{x}{11}}$, where $x$ is the power in watts and 11 is the resistance in ohms.

**A.** Describe the transformations of the parent function $f(x) = \sqrt{x}$ used to graph $I(x)$. expanded horizontally

**B.** The resistance of a lamp is 15 ohms. Write a function to describe the current flowing through the lamp. $I(x) = \sqrt{\frac{x}{15}}$

**C.** Graph the resistance for the DVD player and the lamp on the same graphing calculator screen. See margin.





## Differentiated Instruction AL OL BL

**Interpersonal Learners** Have students work in groups to determine whether families of functions have the same symmetries as the parent function. Encourage students to use computers or graphing calculators to test their conjectures.

Another nonridgid transformation involves absolute value.

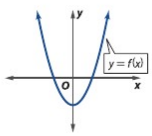
**TechnologyTip**

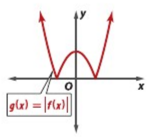
**Absolute Value Transformations**
You can check your graph of an absolute value transformation by using your graphing calculator. You can also graph both functions on the same coordinate axes.

**KeyConcept** Transformations with Absolute Value

**$g(x) = |f(x)|$**

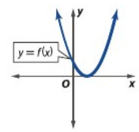
This transformation reflects any portion of the graph of $f(x)$ that is below the $x$-axis so that it is above the $x$-axis.

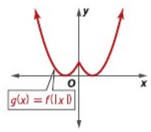



**$g(x) = f(|x|)$**

This transformation results in the portion of the graph of $f(x)$ that is to the left of the $y$-axis being replaced by a reflection of the portion to the right of the $y$-axis.



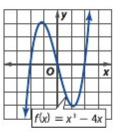



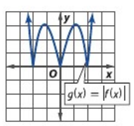

Figure 11.5.6

**Example 7** Describe and Graph Transformations

**Use the graph of $f(x) = x^3 - 4x$ in Figure 11.5.6 to graph each function.**

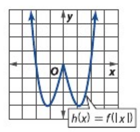
**a.** $g(x) = |f(x)|$

The graph of $f(x)$ is below the $x$-axis on the intervals $(-\infty, -2)$ and $(0, 2)$, so reflect those portions of the graph in the $x$-axis and leave the rest unchanged.




**b.** $h(x) = f(|x|)$

Replace the graph of $f(x)$ to the left of the $y$-axis with a reflection of the graph to the right of the $y$-axis.

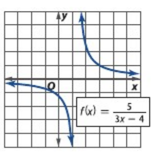



**GuidedPractice**

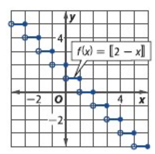
**7A–B. See Chapter 11 Answer Appendix.**

**Use the graph of $f(x)$ shown to graph $g(x) = |f(x)|$ and $h(x) = f(|x|)$.**

**7A.**




**7B.**

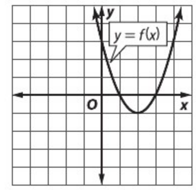




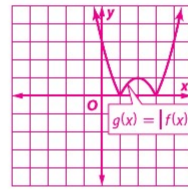



**Additional Example**

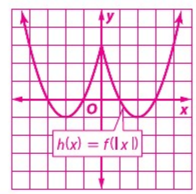
**7** Use the graph of $f(x) = x^2 - 4x + 3$ to graph each function.




**a.** $g(x) = |f(x)|$

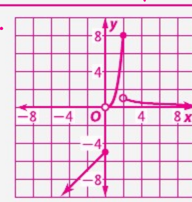



**b.** $h(x) = f(|x|)$

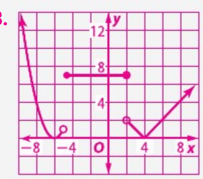



**Additional Answers (Guided Practice)**

**5A.**




**5B.**




**6C.**

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## 3 Practice

### Formative Assessment

Use Exercises 1–46 to check for understanding.

Then use the table below to customize your assignments for students.

### Additional Answers

**1.** D = $\{x \mid x \in \mathbb{R}\}$, R = $\{y \mid y \in \mathbb{Z}\}$. The graph has a $y$-intercept at (0, 0) and $x$-intercepts for $\{x \mid 0 \leq x < 1, x \in \mathbb{R}\}$. The graph has no symmetry. The graph has a jump discontinuity for $\{x \mid x \in \mathbb{Z}\}$. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$. The graph is constant for $\{x \mid x \notin \mathbb{Z}\}$. The graph increases for $\{x \mid x \in \mathbb{Z}\}$.

**2.** D = $\{x \mid x \neq 0, x \in \mathbb{R}\}$, R = $\{y \mid y \neq 0, y \in \mathbb{R}\}$. The graph has no intercepts. The graph is symmetric with respect to the origin. The graph has an infinite discontinuity at $x = 0$. $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to \infty} f(x) = 0$. The graph is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

**3.** D = $\{x \mid x \in \mathbb{R}\}$, R = $\{y \mid y \in \mathbb{R}\}$. The graph has an intercept at (0, 0). The graph is symmetric with respect to the origin. Therefore, it is odd. The graph is continuous. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$. The graph is increasing on $(-\infty, \infty)$.

**4.** D = $\{x \mid x \in \mathbb{R}\}$, R = $\{y \mid y \geq 0, y \in \mathbb{R}\}$. The graph has an intercept at (0, 0). The graph is symmetric with respect to the $y$-axis. The graph is continuous. $\lim\limits_{x \to -\infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$. The graph is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

## Exercises

Describe the following characteristics of the graph of each parent function: domain, range, intercepts, symmetry, continuity, end behavior, and intervals on which the graph is increasing/decreasing. (Example 1) **1–6. See margin.**

**1.** $f(x) = [\![x]\!]$ **2.** $f(x) = \frac{1}{x}$ **3.** $f(x) = x^3$

**4.** $f(x) = x^4$ **5.** $f(x) = c$ **6.** $f(x) = x$

Use the graph of $f(x) = \sqrt{x}$ to graph each function. (Example 2)

**7.** $g(x) = \sqrt{x - 4}$ **8.** $g(x) = \sqrt{x + 3}$

**9.** $g(x) = \sqrt{x + 6} - 4$ **10.** $g(x) = \sqrt{x - 7} + 3$

**7–10. See Chapter 11 Answer Appendix.**

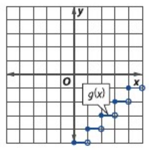
Use the graph of $f(x) = \frac{1}{x}$ to graph each function. (Example 2)

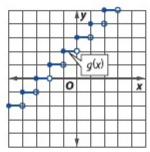
**11.** $g(x) = \frac{1}{x} + 4$ **12.** $g(x) = \frac{1}{x} - 6$

**13.** $g(x) = \frac{1}{x - 6} + 8$ **14.** $g(x) = \frac{1}{x + 7} - 4$

**11–14. See Chapter 11 Answer Appendix.**

Describe how the graphs of $f(x) = [\![x]\!]$ and $g(x)$ are related. Then write an equation for $g(x)$. (Example 3)

**15.**



**16.**
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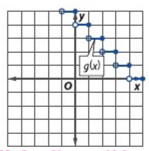
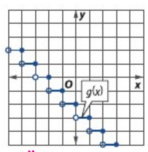
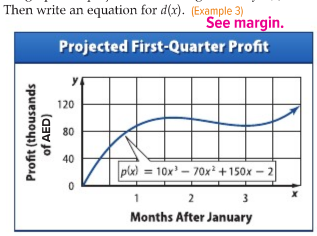



**18.**


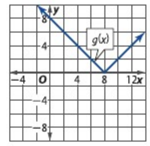

**15–18. See Chapter 11 Answer Appendix.**

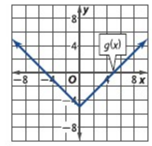
**19. PROFIT** An automobile company experienced an unexpected two-month delay on manufacturing of a new car. The projected profit of the car sales before the delay $p(x)$ is shown below. Describe how the graph of $p(x)$ and the graph of a projection including the delay $d(x)$ are related. Then write an equation for $d(x)$. (Example 3) **See margin.**

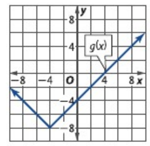


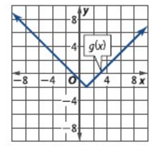

**20–23. See Chapter 11 Answer Appendix.**

Describe how the graphs of $f(x) = |x|$ and $g(x)$ are related. Then write an equation for $g(x)$. (Example 3)

**20.**



**21.**



**22.**



**23.**



**24–31. See Chapter 11 Answer Appendix.**

Identify the parent function $f(x)$ of $g(x)$, and describe how the graphs of $g(x)$ and $f(x)$ are related. Then graph $f(x)$ and $g(x)$ on the same axes. (Example 4)

**24.** $g(x) = 3|x| - 4$ **25.** $g(x) = 3\sqrt{x + 8}$

**26.** $g(x) = \frac{4}{x + 1}$ **27.** $g(x) = 2[\![x - 6]\!]$

**28.** $g(x) = -5[\![x - 2]\!]$ **29.** $g(x) = -2|x + 5|$

**30.** $g(x) = \frac{1}{6x} + 7$ **31.** $g(x) = \frac{\sqrt{x + 3}}{4}$

**32–37. See Chapter 11 Answer Appendix.**

Graph each function. (Example 5)

**32.** $f(x) = \begin{cases} -x^2 & \text{if } x < -2 \\ 3 & \text{if } -2 \leq x < 7 \\ (x - 5)^2 + 2 & \text{if } x \geq 7 \end{cases}$

**33.** $g(x) = \begin{cases} x + 4 & \text{if } x < -6 \\ \frac{1}{x} & \text{if } -6 \leq x < 4 \\ 6 & \text{if } x \geq 4 \end{cases}$

**34.** $f(x) = \begin{cases} 4 & \text{if } x < -5 \\ x^3 & \text{if } -2 \leq x \leq 2 \\ \sqrt{x + 3} & \text{if } x > 3 \end{cases}$

**35.** $h(x) = \begin{cases} |x - 5| & \text{if } x < -3 \\ 4x - 3 & \text{if } -1 \leq x < 3 \\ \sqrt{x} & \text{if } x \geq 4 \end{cases}$

**36.** $g(x) = \begin{cases} 2 & \text{if } x < -4 \\ x^4 - 3x^3 + 5 & \text{if } -1 \leq x < 1 \\ [\![x]\!] + 1 & \text{if } x \geq 3 \end{cases}$

**37.** $f(x) = \begin{cases} -3x - 1 & \text{if } x \leq -1 \\ 0.5x + 5 & \text{if } -1 < x \leq 3 \\ -|x - 5| + 3 & \text{if } x > 3 \end{cases}$





## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
| --- | --- | --- | --- |
| AL Approaching Level | 1–46, 73–78, 80–96 | 1–45 odd, 93–96 | 2–46 even, 73–78, 80–92 |
| OL On Level | 1–45 odd, 47, 48, 49–71 odd, 72–78, 80–96 | 1–46, 93–96 | 47–78, 80–92 |
| BL Beyond Level | 47–96 | | |

748 | Lesson 11-5 | Parent Functions and Transformations

38. **POSTAGE** The cost of a first-class postage stamp in the U.S. from 1988 to 2008 is shown in the table below. Use the data to graph a step function. (Example 5) **See margin.**

| Year | Price (¢) |
|---|---|
| 1988 | 25 |
| 1991 | 29 |
| 1995 | 32 |
| 1999 | 33 |
| 2001 | 34 |
| 2002 | 37 |
| 2006 | 39 |
| 2007 | 41 |
| 2008 | 42 |

**39a. The graph of $c(x)$ is the graph of $f(x)$ compressed vertically and translated 1.99 units up.**

39. **BUSINESS** A no-contract cell phone company charges a flat rate for daily access and AED 0.10 for each minute. The cost of the plan can be modeled by $c(x) = 0.1[\![x]\!] + 1.99$, where $x$ is the number of minutes used. (Example 6)

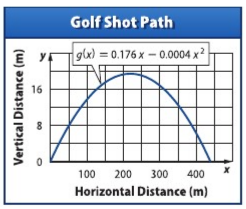
a. Describe the transformation(s) of the parent function $f(x) = [\![x]\!]$ used to graph $c(x)$.

b. The company offers another plan in which the daily access rate is AED 2.49, and the per-minute rate is AED 0.05. What function $d(x)$ can be used to describe the second plan? **$d(x) = 0.05[\![x]\!] + 2.49$**

c. Graph both functions on the same graphing calculator screen. **See margin.**

d. Would the cost of the plans ever equal each other? If so, at how many minutes? **Yes; the plans will equal each other at 10 minutes.**

40. **GOLF** The path of a golf shot can be modeled by the function shown, where $g(x)$ is the vertical distance in meters of the ball from the ground and $x$ is the horizontal distance in meters such that $x = 0$ corresponds to the initial point. (Example 6) **a–d. See margin.**




a. Describe the transformation(s) of the parent function $f(x) = x^2$ used to graph $g(x)$.

b. If a second golfer hits a similar shot 30 m farther down the fairway from the first player, what function $h(x)$ can be used to describe the second golfer's shot?

c. Graph both golfers' shots on the same graphing calculator screen.

d. At what horizontal and vertical distances do the paths of the two shots cross each other?

**Use the graph of $f(x)$ to graph $g(x) = |f(x)|$ and $h(x) = f(|x|)$.** (Example 7) **41–46. See Chapter 11 Answer Appendix.**

41. $f(x) = \frac{2}{x}$

42. $f(x) = \sqrt{x - 4}$

43. $f(x) = x^4 - x^3 - 4x^2$

44. $f(x) = \frac{1}{2}x^3 + 2x^2 - 8x - 2$

45. $f(x) = \frac{1}{x - 3} + 5$

46. $f(x) = \sqrt{x + 2} - 6$

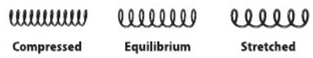
47. **TRANSPORTATION** An example of the standard cost for taxi fare is shown. One unit is equal to a distance of 0.2 km or a time of 60 seconds when the car is not in motion.


a. Write a greatest integer function $f(x)$ that would represent the cost for units of cab fare, where $x > 0$. Round to the nearest unit. **a–c. See Chapter 11 Answer Appendix.**

b. Graph the function.

c. How would the graph of $f(x)$ change if the fare for the first unit increased to AED 3.70 while the cost per unit remained at AED 0.40? Graph the new function.

48. **PHYSICS** The potential energy in joules of a spring that has been stretched or compressed is given by $p(x) = \frac{cx^2}{2}$, where $c$ is the spring constant and $x$ is the distance from equilibrium. When $x$ is negative, the spring is compressed, and when $x$ is positive, the spring is stretched.




a. Describe the transformation(s) of the parent function $f(x) = x^2$ used to graph $p(x)$. **vertical expansion**

b. The graph of the potential energy for a second spring passes through the point (3, 315). Find the spring constant for the spring and write the function for the potential energy. **70; $p(x) = 35x^2$**

**Write and graph the function with the given parent function and characteristics.**

49. $f(x) = \frac{1}{x}$; expanded vertically by a factor of 2; translated 7 units to the left and 5 units up

50. $f(x) = [\![x]\!]$; expanded vertically by a factor of 3; reflected in the $x$-axis; translated 4 units down

**49–54. See Chapter 11 Answer Appendix.**

**PHYSICS** **The distance an object travels as a function of time is given by $f(t) = \frac{1}{2}at^2 + v_0t + x_0$, where $a$ is the acceleration, $v_0$ is the initial velocity, and $x_0$ is the initial position of the object. Describe the transformations of the parent function $f(t) = t^2$ used to graph $f(t)$ for each of the following.**

51. $a = 2, v_0 = 2, x_0 = 0$

52. $a = 2, v_0 = 0, x_0 = 10$

53. $a = 4, v_0 = 8, x_0 = 1$

54. $a = 3, v_0 = 5, x_0 = 3$





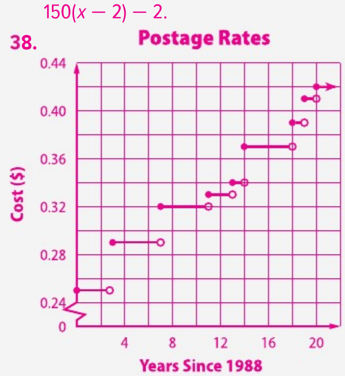
## WatchOut!

**Common Error** In Exercises 41–46, students may have difficulty keeping track of the changes made by the absolute values. Suggest that students first graph the function without absolute values. Then they can reflect parts of the function in the appropriate axis.

## Additional Answers

19. The graph of $d(x)$ is the graph of $p(x)$ translated 2 units to the right; $d(x) = 10(x - 2)^3 - 70(x - 2)^2 + 150(x - 2) - 2$.

38.



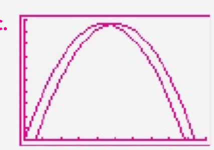

39c.


[0, 20] scl: 2 by [0, 5] scl: 0.5

40a. The graph of $g(x)$ is the graph of $f(x)$ translated 220 units to the right, compressed vertically, reflected in the $x$-axis, and translated 19.36 units up.

40b. $h(x) = -0.0004(x - 250)^2 + 19.36$

40c.



[0, 500] scl: 50 by [0, 20] scl: 2

40d. The shots will cross paths at a horizontal distance of 235 m and a vertical distance of 19.27 m.



## Additional Answers

5. D = $\{x \mid x \in \mathbb{R}\}$, R = $\{y \mid y = c, c \in \mathbb{R}\}$. If $c = 0$, all real numbers are $x$-intercepts. If $c \neq 0$, there are no $x$-intercepts. The graph has a $y$-intercept at $(0, c)$. If $c \neq 0$, the graph is symmetric with respect to the $y$-axis. If $c = 0$, the graph is symmetric with respect to the $x$-axis, $y$-axis, and origin. The graph is continuous. $\lim_{x \to -\infty} f(x) = c$ and $\lim_{x \to \infty} f(x) = c$. The graph is constant on $(-\infty, \infty)$.

6. D = $\{x \mid x \in \mathbb{R}\}$, R = $\{y \mid y \in \mathbb{R}\}$. The graph has an intercept at (0, 0). The graph is symmetric with respect to the origin. The graph is continuous. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$. The graph is increasing on $(-\infty, \infty)$.

## Additional Answers

**72b.** Sample answer: $h(x)$ is the sum of $f(x)$ and $g(x)$.

**72c.** $h(x) = f(x) + g(x)$

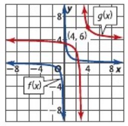
$x^2 + 6x + 10 \stackrel{?}{=} x^2 + 2x + 7 + 4x + 3$

$x^2 + 6x + 10 = x^2 + 6x + 10$

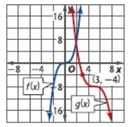
**73.** Sample answer: Both; for the greatest integer function, a shift of $a$ units left is identical to a shift of $a$ units up.

**Write an equation for each $g(x)$.**

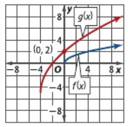
**55.** $g(x) = \frac{2}{x-3} + 4$




**56.** $g(x) = -0.5(x-5)^3 - 8$

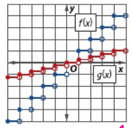



**57.**




$g(x) = 4\sqrt{x+4} - 6$

**58.**




Sample answer: $g(x) = \frac{1}{4}[\![x]\!]$

**59. SHOPPING** The management of a new shopping mall originally predicted that attendance in thousands would follow $f(x) = \sqrt{7x}$ for the first 60 days of operation, where $x$ is the number of days after opening and $x = 1$ corresponds with opening day. Write $g(x)$ in terms of $f(x)$ for each situation below.

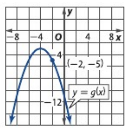
a. Attendance was consistently 12% higher than expected. $g(x) = 1.12f(x)$

b. The opening was delayed 30 days due to construction. **b. See Chapter 11 Answer Appendix.**

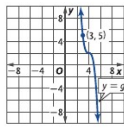
c. Attendance was consistently 450 less than expected. **c.** $g(x) = f(x) - 0.45$

**Identify the parent function $f(x)$ of $g(x)$, and describe the transformation of $f(x)$ used to graph $g(x)$.**

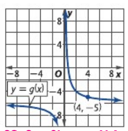
**60.**



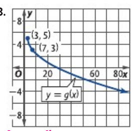

**61.**




**62.**

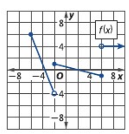



**63.**




**60–63. See Chapter 11 Answer Appendix.**

**Use $f(x)$ to graph $g(x)$.**




**64.** $g(x) = 0.25f(x) + 4$

**65.** $g(x) = 3f(x) - 6$

**66.** $g(x) = f(x-5) + 3$

**67.** $g(x) = -2f(x) + 1$

**64–67. See Chapter 11 Answer Appendix.**

**Use $f(x) = \frac{8}{\sqrt{x+6}} - 4$ to graph each function.**

**68.** $g(x) = 2f(x) + 5$

**69.** $g(x) = -3f(x) + 6$

**70.** $g(x) = f(4x) - 5$

**71.** $g(x) = f(2x+1) + 8$

**68–71. See Chapter 11 Answer Appendix.**

**72. MULTIPLE REPRESENTATIONS** In this problem, you will investigate operations with functions. Consider

- $f(x) = x^2 + 2x + 7$,
- $g(x) = 4x + 3$, and
- $h(x) = x^2 + 6x + 10$.

a. **TABULAR** Copy and complete the table below for three values for $a$.

| $a$ | $f(a)$ | $g(a)$ | $f(a) + g(a)$ | $h(a)$ |
|---|---|---|---|---|
| 3 | 22 | 15 | 37 | 37 |
| −4 | 15 | −13 | 2 | 2 |
| 15 | 262 | 63 | 325 | 325 |

b. **VERBAL** How are $f(x)$, $g(x)$, and $h(x)$ related?

c. **ALGEBRAIC** Prove the relationship from part **b** algebraically. **b–c. See margin.**

## H.O.T. Problems Use Higher-Order Thinking Skills

**73. ERROR ANALYSIS** Obaid and Majed are describing the transformation $g(x) = [\![x + 4]\!]$. Obaid says that the graph is shifted 4 units to the left, while Majed says that the graph is shifted 4 units up. Is either of them correct? Explain. **See margin.**

**74. REASONING** Let $f(x)$ be an odd function. If $g(x)$ is a reflection of $f(x)$ in the $x$-axis and $h(x)$ is a reflection of $g(x)$ in the $y$-axis, what is the relationship between $f(x)$ and $h(x)$? Explain. **See Chapter 11 Answer Appendix.**

**75. WRITING IN MATH** Explain why order is important when transforming a function with reflections and translations. **See Chapter 11 Answer Appendix.**

**REASONING** **Determine whether the following statements are *sometimes*, *always*, or *never* true. Explain your reasoning.**

**76.** If $f(x)$ is an even function, then $f(x) = |f(x)|$.

**77.** If $f(x)$ is an odd function, then $f(-x) = -|f(x)|$.

**78.** If $f(x)$ is an even function, then $f(-x) = -|f(x)|$.

**76–78. See Chapter 11 Answer Appendix.**

**79. CHALLENGE** Describe the transformation of $f(x) = \sqrt{x}$ if $(-2, -6)$ lies on the curve. **See Chapter 11 Answer Appendix.**

**80. REASONING** Suppose $(a, b)$ is a point on the graph of $f(x)$. Describe the difference between the transformations of $(a, b)$ when the graph of $f(x)$ is expanded vertically by a factor of 4 and when the graph of $f(x)$ is compressed horizontally by a factor of 4. **See Chapter 11 Answer Appendix.**

**81. WRITING IN MATH** Use words, graphs, tables, and equations to relate parent functions and transformations. Show this relationship through a specific example. **See students' work.**





750 | Lesson 11-5 | Parent Functions and Transformations

## Spiral Review

Find the average rate of change of each function on the given interval. (Lesson 11-4)

**82.** $g(x) = -2x^2 + x - 3$; $[-1, 3]$ **−3**

**83.** $g(x) = x^2 - 6x + 1$; $[4, 8]$ **6**

**84.** $f(x) = -2x^3 - x^2 + x - 4$; $[-2, 3]$ **−14**

Use the graph of each function to describe its end behavior. Support the conjecture numerically. (Lesson 11-3) **85–87. See margin.**

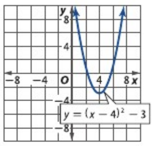
**85.** $q(x) = -\frac{12}{x}$

**86.** $f(x) = \frac{0.5}{x^2}$

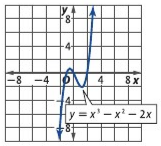
**87.** $p(x) = \frac{x+2}{x-3}$

Use the graph of each function to estimate its $y$-intercept and zero(s). Then find these values algebraically. (Lesson 11-2) **88–90. See margin.**

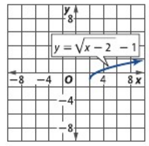
**88.**




**89.**




**90.**



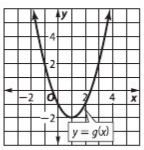

**91.** **GOVERNMENT** The number of times each of the first 42 presidents vetoed bills are listed below. What is the standard deviation of the data? **about 118.60**

2, 0, 0, 7, 1, 0, 12, 1, 0, 10, 3, 0, 0, 9,
7, 6, 29, 93, 13, 0, 12, 414, 44, 170, 42, 82, 39, 44,
6, 50, 37, 635, 250, 181, 21, 30, 43, 66, 31, 78, 44, 25

**92.** **COMPETITION** In a competition the player must guess which five of the white balls numbered from 1 to 49 will be chosen. The order in which the balls are chosen does not matter. The player must also guess which one of the red balls numbered from 1 to 42 will be chosen. How many ways can the player predict the balls that will be chosen. **80,089,128**

## Skills Review for Standardized Tests

**93.** **SAT/ACT** The figure shows the graph of $y = g(x)$, which has a minimum located at $(1, -2)$. What is the maximum value of $h(x) = -3g(x) - 1$? **B**




A 6

B 5

C 3

D 2

E It cannot be determined from the information given.

**94.** **REVIEW** What is the simplified form of $\frac{4x^3y^2z^{-1}}{(x^{-2}y^3z^2)^2}$? $\frac{4x^7}{y^4z^5}$

**95.** What is the range of $y = \frac{x^2+8}{2}$? **G**

F $\{y \mid y \neq \pm 2\sqrt{2}\}$

G $\{y \mid y \geq 4\}$

H $\{y \mid y \geq 0\}$

J $\{y \mid y \leq 0\}$

**96.** **REVIEW** What is the effect on the graph of $y = kx^2$ as $k$ decreases from 3 to 2? **D**

A The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the $y$-axis.

B The graph is rotated 90° about the origin.

C The graph becomes narrower.

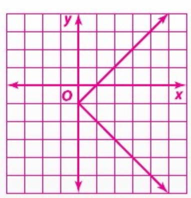
D The graph becomes wider.





## Differentiated Instruction BL

**Extension** Graph $x = |y + 1|$. Describe how it is a transformation of the parent function $y = |x|$.
The graph of $x = |y + 1|$ is the graph of $y = |x|$ rotated 90° clockwise about the origin, and translated 1 unit down.



### WatchOut!

**Error Analysis** In Exercise 73, both interpretations are correct. Ask students to verify this by first translating the parent function 4 units up and then translating the parent function 4 units to the right. They will see the result of both steps is the same.

## 4 Assess

**Ticket Out the Door** Have students describe how the graph of $g(x) = -\frac{1}{4}(x + 3)^2 + 4$ is related to its parent function. It is the graph of $f(x) = x^2$, translated 3 units left, compressed by a factor of one-fourth, reflected in the $x$-axis, and translated 4 units up.

### Additional Answers

**85.** 0; Sample answer: As $x \to \infty$, the denominator of the fraction will increase and the value of the fraction will approach 0, so $q(x)$ will approach 0.

**86.** 0; Sample answer: As $x \to \infty$, the denominator of the fraction will increase and the value of the fraction will approach 0, so $f(x)$ will approach 0.

**87.** 1; Sample answer: As $x \to \infty$, the fraction will get closer and closer to $\frac{x}{x}$, so $p(x)$ will approach 1.

**88.** $y$-intercept: 13; zeros: 5.73, 2.27;
$$(x-4)^2 - 3 = 0$$
$$(x-4)^2 = 3$$
$$x - 4 = \pm\sqrt{3}$$
$$x = 4 \pm \sqrt{3}$$
$$x \approx 5.73 \text{ or } x \approx 2.27$$

**89.** $y$-intercept: 0; zeros: −1, 0, 2;
$$x^3 - x^2 - 2x = 0$$
$$x(x^2 - x - 2) = 0$$
$$x(x-2)(x+1) = 0$$
$$x = 0 \text{ or } x - 2 = 0 \text{ or } x + 1 = 0$$
$$x = 2 \qquad x = -1$$

**90.** no $y$-intercept; zero: 3;
$$\sqrt{x-2} - 1 = 0$$
$$\sqrt{x-2} = 1$$
$$x - 2 = 1$$
$$x = 3$$

## 1 Focus

**Objective** Use a graphing calculator to solve nonlinear inequalities.

### Teaching Tip

Before students turn on their calculators, explain the first step in the Activity. The first inequality is derived by replacing the right side of the inequality with *y*. The second inequality is derived by replacing the left side of the inequality with *y*.

## 2 Teach

### Working in Cooperative Groups

Pair students who are proficient with a graphing calculator with students who are not. Have each pair take turns entering information into the calculator.

**Practice** Have students complete Exercises 1, 3, 5, 7, and 8.

## 3 Assess

### Formative Assessment

Have students work independently to complete Exercises 2, 4, and 6. Have students draw their graphs onto a sheet of paper, along with solution sets.

### From Concrete to Abstract

Ask students to summarize how to find the solution region of two or more nonlinear inequalities.

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# EXTEND 11-5 Graphing Technology Lab: Nonlinear Inequalities

**Objective**

- Use a graphing calculator to solve nonlinear inequalities.

A nonlinear inequality in one variable can be solved graphically by converting it into two inequalities in two variables and finding the intersection. You can use a graphing calculator to find this intersection.

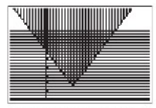
**Activity** Solve an Inequality by Graphing

**Solve $2|x - 4| + 3 < 15$.**

**Step 1** Separate this inequality into two inequalities, one for each side of the inequality symbol. Replace each side with *y* to form the new inequalities.
$2|x - 4| + 3 <$ Y1; Y2 $< 15$

**Step 2** Graph each inequality. Go to the left of the equals sign and select ENTER until the shaded triangles flash to make each inequality sign. The triangle above represents *greater than* and the triangle below represents *less than*. For abs(, press MATH ▶ 1.


**Step 3** Graph the inequalities in the appropriate window. Either use the ZOOM feature or adjust the window manually to display both graphs. Any window that shows the two intersection points will work.



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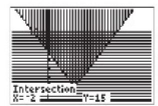
**Step 4** The darkly shaded area indicates the intersection of the graphs and the solution of the system of inequalities. Use the intersection feature to find that the two graphs intersect at (−2, 15) and (10, 15).

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**Step 5** The solution occurs in the region of the graph where $-2 < x < 10$. Thus, the solution to $2|x - 4| + 3 < 15$ is the set of *x*-values such that $-2 < x < 10$. Check your solution algebraically by confirming that an *x*-value in this interval is a solution of the inequality.

**StudyTip**

**Adjusting the Window** You can use the ZoomFit or ZoomOut options or manually adjust the window to include both graphs.

### Exercises

**Solve each inequality by graphing.**

1. $3|x + 2| - 4 > 8$ **$(-\infty, -6) \cup (2, \infty)$**
2. $-2|x + 4| + 6 \leq 2$ **$(-\infty, -6] \cup [-2, \infty)$**
3. $5|2x + 1| > 15$ **$(-\infty, -2) \cup (1, \infty)$**
4. $-3|2x - 3| + 1 \leq 10$ **$(-\infty, \infty)$**
5. $|x - 6| > x + 2$ **$(-\infty, -2)$**
6. $|2x + 1| \geq 4x - 3$ **$(-\infty, 2]$**

### Extension

7. **REASONING** Describe the appearance of the graph for an inequality with no solution.
8. **CHALLENGE** Solve $-10x - 32 < |x + 3| - 2 < -|x + 4| + 8$ by graphing. **$(-3, 1.5)$**

**7. Sample answer: While each individual inequality will have a shaded area, there will be no intersection of the two shaded areas.**

# LESSON 11-6 Function Operations and Composition of Functions

**Then**

- You evaluated functions. (Lesson 11-1)

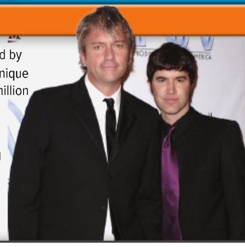
**Now**

1. Perform operations with functions.
2. Find compositions of functions.

**Why?**

- In April 2008, the top social networking site, founded by C. DeWolfe and T. Anderson, had over 60.4 million unique visitors. The number two site at that time had 24.9 million unique visitors, or 35.5 million fewer visitors.

Suppose $A(t)$ and $B(t)$ model the number of unique visitors to the number one and two social networking sites, respectively, $t$ years since 2000. $A(t) - B(t)$ represents the difference in the number of unique visitors between the two sites for $t$ years after 2000.



**New Vocabulary**

composition

## 1 Operations with Functions

Just as you can combine two real numbers using addition, subtraction, multiplication, and division, you can combine two functions.

### Key Concept Operations with Functions

Let $f$ and $g$ be two functions with intersecting domains. Then for all $x$-values in the intersection, the sum, product, difference, and quotient of $f$ and $g$ are new functions defined as follows.

| | | | |
|---|---|---|---|
| **Sum** | $(f + g)(x) = f(x) + g(x)$ | **Product** | $(f \cdot g)(x) = f(x) \cdot g(x)$ |
| **Difference** | $(f - g)(x) = f(x) - g(x)$ | **Quotient** | $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$ |

For each new function, the domain consists of those values of $x$ common to the domains of $f$ and $g$. The domain of the quotient function is further restricted by excluding any values that make the denominator, $g(x)$, zero.

### Example 1 Operations with Functions

**Given $f(x) = x^2 + 4x$, $g(x) = \sqrt{x + 2}$, and $h(x) = 3x - 5$, find each function and its domain.**

**a.** $(f + g)(x)$

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= (x^2 + 4x) + \left(\sqrt{x + 2}\right)\\ &= x^2 + 4x + \sqrt{x + 2}\end{aligned}$$

The domain of $f$ is $(-\infty, \infty)$, and the domain of $g$ is $[-2, \infty)$. So, the domain of $(f + g)$ is the intersection of these domains or $[-2, \infty)$.

**b.** $(f - h)(x)$

$$\begin{aligned}(f - h)(x) &= f(x) - h(x)\\ &= (x^2 + 4x) - (3x - 5)\\ &= x^2 + 4x - 3x + 5\\ &= x^2 + x + 5\end{aligned}$$

The domains of $f$ and $h$ are both $(-\infty, \infty)$, so the domain of $(f - h)$ is $(-\infty, \infty)$.

**c.** $(f \cdot h)(x)$

$$\begin{aligned}(f \cdot h)(x) &= f(x) \cdot h(x)\\ &= (x^2 + 4x)(3x - 5)\\ &= 3x^3 - 5x^2 + 12x^2 - 20x\\ &= 3x^3 + 7x^2 - 20x\end{aligned}$$

The domains of $f$ and $h$ are both $(-\infty, \infty)$, so the domain of $(f \cdot h)$ is $(-\infty, \infty)$.

**d.** $\left(\frac{h}{f}\right)(x)$

$$\left(\frac{h}{f}\right)(x) = \frac{h(x)}{f(x)} \text{ or } \frac{3x - 5}{x^2 + 4x}$$

The domain of $h$ and $f$ are both $(-\infty, \infty)$, but $x = 0$ or $x = -4$ yields a zero in the denominator of $\left(\frac{h}{f}\right)$. So, the domain of $\left(\frac{h}{f}\right)$ is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

753



## 1 Focus

### Vertical Alignment

**Before Lesson 11-6** Evaluate functions.

**Lesson 11-6** Perform operations with functions. Find compositions of functions.

**After Lesson 11-6** Find inverses of relations and functions algebraically and graphically.

## 2 Teach

### Scaffolding Questions

Have students read the **Why?** section of the lesson.

**Ask:**

- Depending on the month, two products sell at different rates. How can we compare the rates? Subtract the sales for one product from the sales for the other.
- A company is interested in the monthly ratio of homes sold to homes for sale. What expression represents this ratio? Divide the homes sold by the homes for sale.

## 1 Operations with Functions

**Example 1** shows how to add, subtract, multiply, and divide functions.

### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

### Additional Example

**1** Given $f(x) = x^2 - 2x$, $g(x) = 3x - 4$, and $h(x) = -2x^2 + 1$, find each function and its domain.

**a.** $(f + g)(x)$ $(f + g)(x) = x^2 + x - 4$; $D = (-\infty, \infty)$

**b.** $(f - h)(x)$ $(f - h)(x) = 3x^2 - 2x - 1$; $D = (-\infty, \infty)$

**c.** $(f \cdot g)(x)$ $(f \cdot g)(x) = 3x^3 - 10x^2 + 8x$; $D = (-\infty, \infty)$

**d.** $\left(\frac{h}{f}\right)(x)$ $\left(\frac{h}{f}\right)(x) = \frac{-2x^2 + 1}{x^2 - 2x}$; $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

## 2 Composition of Functions

**Examples 2 and 3** show how to compose functions and find a composite function with a restricted domain. **Example 4** shows how to decompose a composite function. **Example 5** shows how to use a composite function.

### Guided Practice

**Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. State the domain of each new function.** 1A–B. See margin.

**1A.** $f(x) = x - 4$, $g(x) = \sqrt{9 - x^2}$

**1B.** $f(x) = x^2 - 6x - 8$, $g(x) = \sqrt{x}$

## 2 Composition of Functions

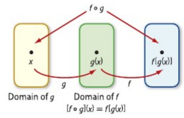
The function $y = (x - 3)^2$ combines the linear function $y = x - 3$ with the squaring function $y = x^2$, but the combination does not involve addition, subtraction, multiplication, or division. This combining of functions, called *composition*, is the result of one function being used to evaluate a second function.

### KeyConcept Composition of Functions

The composition of function $f$ with function $g$ is defined by

$$[f \circ g](x) = f[g(x)].$$

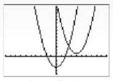
The domain of $f \circ g$ includes all $x$-values in the domain of $g$ that map to $g(x)$-values in the domain of $g$ as shown.




In the composition $f \circ g$, which is read as *f composition g* or *f of g*, the function $g$ is applied first and then $f$.

### WatchOut!

**Order of Composition** In most cases, $g \circ f$ and $f \circ g$ are different functions. That is, composition of functions is not commutative. Notice that the graphs of $[f \circ g](x) = x^2 - 8x + 17$ and $[g \circ f](x) = x^2 - 3$ from Example 2 are different.



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### Example 2 Compose Two Functions

**Given $f(x) = x^2 + 1$ and $g(x) = x - 4$, find each of the following.**

**a.** $[f \circ g](x)$

| | |
|---|---|
| $[f \circ g](x) = f[g(x)]$ | Definition of $f \circ g$ |
| $= f(x - 4)$ | Replace $g(x)$ with $x - 4$. |
| $= (x - 4)^2 + 1$ | Substitute $x - 4$ for $x$ in $f(x)$. |
| $= x^2 - 8x + 16 + 1$ or $x^2 - 8x + 17$ | Simplify. |

**b.** $[g \circ f](x)$

| | |
|---|---|
| $[g \circ f](x) = g[f(x)]$ | Definition of $g \circ f$ |
| $= g(x^2 + 1)$ | Replace $f(x)$ with $x^2 + 1$. |
| $= (x^2 + 1) - 4$ or $x^2 - 3$ | Substitute $x^2 + 1$ for $x$ in $g(x)$. |

**c.** $[f \circ g](2)$

Evaluate the expression $[f \circ g](x)$ you wrote in part **a** for $x = 2$.

$[f \circ g](2) = (2)^2 - 8(2) + 17$ or 5 Substitute 2 for $x$ in $x^2 - 8x + 17$.

### Guided Practice

**For each pair of functions, find $[f \circ g](x)$, $[g \circ f](x)$, and $[f \circ g](3)$.**

**2A.** $f(x) = 3x + 1$, $g(x) = 5 - x^2$
$-3x^2 + 16$, $-9x^2 - 6x + 4$, $-11$

**2B.** $f(x) = 6x^2 - 4$, $g(x) = x + 2$
$6x^2 + 24x + 20$, $6x^2 - 2$, $146$





### Additional Answers (Guided Practice)

**1A.** $(f + g)(x) = x - 4 + \sqrt{9 - x^2}$, $D = [-3, 3]$; $(f - g)(x) = x - 4 - \sqrt{9 - x^2}$, $D = [-3, 3]$; $(f \cdot g)(x) = x\sqrt{9 - x^2} - 4\sqrt{9 - x^2}$, $D = [-3, 3]$; $\left(\frac{f}{g}\right)(x) = \frac{x - 4}{\sqrt{9 - x^2}}$, $D = (-3, 3)$

**1B.** $(f + g)(x) = x^2 - 6x - 8 + \sqrt{x}$, $D = [0, \infty)$; $(f - g)(x) = x^2 - 6x - 8 - \sqrt{x}$, $D = [0, \infty)$; $(f \cdot g)(x) = x^2\sqrt{x} - 6x\sqrt{x} - 8\sqrt{x}$, $D = [0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x^2 - 6x - 8}{\sqrt{x}}$, $D = (0, \infty)$

754 | Lesson 11-6 | Function Operations and Composition of Functions

Because the domains of $f$ and $g$ in Example 2 include all real numbers, the domain of $f \circ g$ is all real numbers, $\mathbb{R}$.

When the domains of $f$ or $g$ are restricted, the domain of $f \circ g$ is restricted to all $x$-values in the domain of $g$ whose range values, $g(x)$, are in the domain of $f$.

**Example 3** Find a Composite Function with a Restricted Domain

**Find $f \circ g$.**

**a.** $f(x) = \dfrac{1}{x+1}$, $g(x) = x^2 - 9$

To find $f \circ g$, you must first be able to find $g(x) = x^2 - 9$, which can be done for all real numbers. Then you must be able to evaluate $f(x) = \dfrac{1}{x+1}$ for each of these $g(x)$-values, which can only be done when $g(x) \neq -1$. Excluding from the domain those values for which $x^2 - 9 = -1$, namely when $x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$, the domain of $f \circ g$ is $\{x \mid x \neq \pm 2\sqrt{2}, x \in \mathbb{R}\}$.

Now find $[f \circ g](x)$.

| | |
|---|---|
| $[f \circ g](x) = f[g(x)]$ | Definition of $f \circ g$ |
| $= f(x^2 - 9)$ | Replace $g(x)$ with $x^2 - 9$. |
| $= \dfrac{1}{x^2 - 9 + 1}$ or $\dfrac{1}{x^2 - 8}$ | Substitute $x^2 - 9$ for $x$ in $f(x)$. |

Notice that $\dfrac{1}{x^2 - 8}$ is undefined when $x^2 - 8 = 0$, which is when $x = \pm 2\sqrt{2}$. Because the implied domain is the same as the domain determined by considering the domains of $f$ and $g$, the composition can be written as $[f \circ g](x) = \dfrac{1}{x^2 - 8}$ for $x \neq \pm 2\sqrt{2}$.

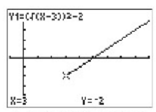
**b.** $f(x) = x^2 - 2$, $g(x) = \sqrt{x - 3}$

To find $f \circ g$, you must first be able to find $g(x)$, which can only be done for $x \geq 3$. Then you must be able to square each of these $g(x)$-values and subtract 2, which can be done for all real numbers. Therefore, the domain of $f \circ g$ is $\{x \mid x \geq 3, x \in \mathbb{R}\}$. Now find $[f \circ g](x)$.

| | |
|---|---|
| $[f \circ g](x) = f[g(x)]$ | Definition of $f \circ g$ |
| $= f(\sqrt{x-3})$ | Replace $g(x)$ with $\sqrt{x-3}$. |
| $= (\sqrt{x-3})^2 - 2$ | Substitute $\sqrt{x-3}$ for $x$ in $f(x)$. |
| $= x - 3 - 2$ or $x - 5$ | Simplify. |

Once the composition is simplified, it appears that the function is defined for all reals, which is known to be untrue. Therefore, write the composition as $[f \circ g](x) = x - 5$ for $x \geq 3$.

**CHECK** Use a graphing calculator to check this result. Enter the function as $y = (\sqrt{x-3})^2 - 2$. The graph appears to be part of the line $y = x - 5$. Then use the TRACE feature to help determine that the domain of the composite function begins at $x = 3$ and extends to $\infty$.

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**Guided Practice**

**3A.** $f(x) = \sqrt{x+1}$, $g(x) = x^2 - 1$

**3B.** $f(x) = \dfrac{5}{x}$, $g(x) = x^2 + x$

An important skill in calculus is to be able to *decompose* a function into two simpler functions. To decompose a function $h$, find two functions with a composition of $h$.

**StudyTip**

**Domains of Composite Functions** It is very important to complete the domain analysis before performing the composition. Domain restrictions may not be evident after the composition is simplified.

**3A.** $[f \circ g](x) = |x|$

**3B.** $[f \circ g](x) = \dfrac{5}{x^2 + x}$ for $x \neq 0$ or $x \neq -1$

**StudyTip**

**Using Absolute Value** Recall that when you find an even root of an even power and the result is an odd power, you must use the absolute value of the result to ensure that the answer is nonnegative. For example, $\sqrt{x^2} = |x|$.





## Additional Examples

**2** Given $f(x) = 2x^2 - 1$ and $g(x) = x + 3$, find each of the following.

**a.** $[f \circ g](x)$ $[f \circ g](x) = 2x^2 + 12x + 17$

**b.** $[g \circ f](x)$ $[g \circ f](x) = 2x^2 + 2$

**c.** $[f \circ g](2)$ $[f \circ g](2) = 49$

**3** Find $f \circ g$.

**a.** $f(x) = \sqrt{x - 1}$, $g(x) = (x - 1)^2$ $[f \circ g](x) = \sqrt{x^2 - 2x}$ for $x \leq 0$ or $x \geq 2$

**b.** $f(x) = \dfrac{1}{x}$, $g(x) = \sqrt{x^2 - 1}$ $[f \circ g](x) = \dfrac{\sqrt{x^2 - 1}}{x^2 - 1}$ for $x < -1$ or $x > 1$

### Focus on Mathematical Content

**Composition of Functions** The composition of functions is not generally commutative. However, there are some pairs of functions where $f[g(x)] = g[f(x)]$. When $f[g(x)] = g[f(x)] = x$, $f$ and $g$ are inverses of each other.

### Teach with Tech

**Spreadsheet** Using a spreadsheet and an example from the lesson, students will work in teams to create a list of input, or domain, values and output, or range, values for the first function. Then have students use the output as input for the composite function. Students should highlight any domain restrictions in the spreadsheet.

## Differentiated Instruction AL OL

**Interpersonal Learners** Have students work in pairs. Each student thinks of a function. The pair works together to find the sum, difference, product, and quotient of the functions and the composites of the two functions.

## Additional Examples

**4** Find two functions $f$ and $g$ such that $h(x) = [f \circ g](x)$. Neither function may be the identity function $f(x) = x$. Sample answers given.

**a.** $h(x) = \dfrac{1}{(x+2)^2}$

$g(x) = x + 2$ and $f(x) = \dfrac{1}{x^2}$

**b.** $h(x) = 3x^2 - 12x + 12$

$g(x) = x - 2$ and $f(x) = 3x^2$

**5** **COMPUTER ANIMATION** An animator starts with an image of a circle with a radius of 25 pixels. The animator then increases the radius by 10 pixels per second.

**a.** Find functions to model the data. $R(t) = 25 + 10t$; $A(R) = \pi R^2$

**b.** Find $A \circ R$. What does the function represent? $[A \circ R](t) = 100\pi t^2 + 500\pi t + 625\pi$; the function models the area of the circle as a function of time.

**c.** How long does it take for the circle to quadruple its original size? 2.5 seconds

### Additional Answers (Guided Practice)

**5B.** $[c \circ d](x) = 0.85x - 100$, $[d \circ c](x) = 0.85x - 85$; $[c \circ d](x)$ represents the price of the computer using the discount and then the coupon and $[d \circ c](x)$ represents the price of the computer using the coupon and then the discount.

**5C.** Sample answer: Using the discount and then the coupon, or $[c \circ d](x)$, results in the lower price. For example, if a student wants to purchase a AED 1000 computer, he or she will pay AED 750 using the discount and then the coupon and AED 765 using the coupon and then the discount.

## Example 4 Decompose a Composite Function

**Find two functions $f$ and $g$ such that $h(x) = [f \circ g](x)$. Neither function may be the identity function $f(x) = x$.**

**a.** $h(x) = \sqrt{x^3 - 4}$

Observe that $h$ is defined using the square root of $x^3 - 4$. So one way to write $h$ as a composition of two functions is to let $g(x) = x^3 - 4$ and $f(x) = \sqrt{x}$. Then

$$h(x) = \sqrt{x^3 - 4} = \sqrt{g(x)} = f[g(x)] \text{ or } [f \circ g](x).$$

**b.** $h(x) = 2x^2 + 20x + 50$

$h(x) = 2x^2 + 20x + 50$ — Notice that $h(x)$ is factorable.

$= 2(x^2 + 10x + 25)$ or $2(x + 5)^2$ — Factor.

One way to write $h(x)$ as a composition is to let $f(x) = 2x^2$ and $g(x) = x + 5$.

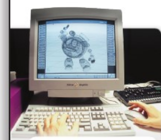
$$h(x) = 2(x + 5)^2 = 2[g(x)]^2 = f[g(x)] \text{ or } [f \circ g](x).$$

### Guided Practice

**4A.** $h(x) = x^2 - 2x + 1$ $g(x) = x - 1$, $f(x) = x^2$

**4B.** $h(x) = \dfrac{1}{x+7}$ $g(x) = x + 7$, $f(x) = \dfrac{1}{x}$

You can use the composition of functions to solve real-world problems.



**Real-World Career**

**Computer Animator**
Animators work in many industries to create the animated images used in movies, television, and video games. Computer animators must be artistic, and most have received post-secondary training at specialized schools.

## Real-World Example 5 Compose Real-World Functions

**COMPUTER ANIMATION To animate the approach of an opponent directly in front of a player, a computer game animator starts with an image of a 20-pixel by 60-pixel rectangle. The animator then increases each dimension of the rectangle by 15 pixels per second.**

**a. Find functions to model the data.**

The length $L$ of the rectangle increases at a rate of 15 pixels per second, so $L(t) = 20 + 15t$, where $t$ is the time in seconds and $t \geq 0$. The area of the rectangle is its length $L$ times its width. The width is 40 pixels more than its length or $L + 40$. So, the area of the rectangle is $A(L) = L(L + 40)$ or $L^2 + 40L$ and $L \geq 20$.

**b. Find $A \circ L$. What does this function represent?**

| | |
|---|---|
| $A \circ L = A[L(t)]$ | Definition of $A \circ L$ |
| $= A(20 + 15t)$ | Replace $L(t)$ with $20 + 15t$. |
| $= (20 + 15t)^2 + 40(20 + 15t)$ | Substitute $20 + 15t$ for $L$ in $A(L)$. |
| $= 225t^2 + 1200t + 1200$ | Simplify. |

This composite function models the area of the rectangle as a function of time.

**c. How long does it take for the rectangle to triple its original size?**

The initial area of the rectangle is $20 \cdot 60$ or 1200 pixels. The rectangle will be three times its original size when $[A \circ L](t) = 225t^2 + 1200t + 1200 = 3600$. Solve for $t$ to find that $t \approx 1.55$ or $-6.88$. Because a negative $t$-value is not part of the domain of $L(t)$, it is also not part of the domain of the composite function. The area will triple after about 1.55 seconds.

### Guided Practice

**5.** **BUSINESS** A computer store offers a 15% discount to college students on the purchase of any notebook computer. The store also advertises AED 100 coupons.

**A.** Find functions to model the data. $d(x) = 0.85x$; $c(x) = x - 100$ **5B–C. See margin.**

**B.** Find $[c \circ d](x)$ and $[d \circ c](x)$. What does each composite function represent?

**C.** Which composition of the coupon and discount results in the lower price? Explain.





## Differentiated Instruction AL

**Kinesthetic Learners** Have students work in groups of two to four. Write integers from –10 to 10 on separate index cards. Ask one student in each group to be the gatekeeper for the first function in a composite function. Other students in the group pass the index cards to the gatekeeper who rejects or accepts each card, depending on whether the number is in the domain of the function. After the review, another student is the gatekeeper for the second function. Students can use this process to define the domain of the composite function.

## Exercises

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. State the domain of each new function. (Example 1) **1–4. See margin.**

1. $f(x) = x^2 + 4$, $g(x) = \sqrt{x}$
2. $f(x) = 8 - x^3$, $g(x) = x - 3$
3. $f(x) = x^2 + 5x + 6$, $g(x) = x + 2$
4. $f(x) = x - 9$, $g(x) = x + 5$
5. $f(x) = x^2 + x$, $g(x) = 9x$
6. $f(x) = x - 7$, $g(x) = x + 7$
7. $f(x) = \frac{6}{x}$, $g(x) = x^3 + x$
8. $f(x) = \frac{x}{4}$, $g(x) = \frac{3}{x}$
9. $f(x) = \frac{1}{\sqrt{x}}$, $g(x) = 4\sqrt{x}$
10. $f(x) = \frac{3}{x}$, $g(x) = x^4$
11. $f(x) = \sqrt{x+8}$, $g(x) = \sqrt{x+5} - 3$
12. $f(x) = \sqrt{x+6}$, $g(x) = \sqrt{x-4}$

**5–12. See Chapter 11 Answer Appendix.**

13. **BUDGETING** Suppose a budget in dirhams for one person for one month is approximated by $f(x) = 25x + 350$ and $g(x) = 15x + 200$, where $f$ is the cost of rent and groceries, $g$ is the cost of gas and all other expenses, and $x = 1$ represents the end of the first week. (Example 1) **a–c. See Chapter 11 Answer Appendix.**
    a. Find $(f+g)(x)$ and the relevant domain.
    b. What does $(f+g)(x)$ represent?
    c. Find $(f+g)(4)$. What does this value represent?

14. **PHYSICS** Two different forces act on an object being pushed across a floor: the force of the person pushing the object and the force of friction. If $W$ is work in joules, $F$ is force in newtons, and $d$ is displacement of the object in meters, $W_p(d) = F_p d$ describes the work of the person and $W_f(d) = F_f d$ describes the work done by friction. The increase in kinetic energy of the object is the difference between the work done by the person $W_p$ and the work done by friction $W_f$. (Example 1)
    a. Find $(W_p - W_f)(d)$. **$d(F_p - F_f)$ or $F_p d - F_f d$**
    b. Determine the net work expended when a person pushes a box 50 m with a force of 95 newtons and friction exerts a force of 55 newtons. **200 joules**

**15–20. See Chapter 11 Answer Appendix.**
For each pair of functions, find $[f \circ g](x)$, $[g \circ f](x)$, and $[f \circ g](6)$. (Example 2)

15. $f(x) = 2x - 3$, $g(x) = 4x - 8$
16. $f(x) = -2x^2 - 5x + 1$, $g(x) = -5x + 6$
17. $f(x) = 8 - x^2$, $g(x) = x^2 + x + 1$
18. $f(x) = x^2 - 16$, $g(x) = x^2 + 7x + 11$
19. $f(x) = 3 - x^2$, $g(x) = x^3 + 1$
20. $f(x) = 2 + x^4$, $g(x) = -x^2$

**21–28. See Chapter 11 Answer Appendix.**
Find $f \circ g$. (Example 3)

21. $f(x) = \frac{1}{x+1}$, $g(x) = x^2 - 4$
22. $f(x) = \frac{2}{x-3}$, $g(x) = x^2 + 6$
23. $f(x) = \sqrt{x+4}$, $g(x) = x^2 - 4$
24. $f(x) = x^2 - 9$, $g(x) = \sqrt{x+3}$
25. $f(x) = \frac{5}{x}$, $g(x) = \sqrt{6-x}$
26. $f(x) = -\frac{4}{x}$, $g(x) = \sqrt{x+8}$
27. $f(x) = \sqrt{x+5}$, $g(x) = x^2 + 4x - 1$
28. $f(x) = \sqrt{x-2}$, $g(x) = x^2 + 8$

29. **RELATIVITY** In the theory of relativity, $m(v) = \frac{100}{\sqrt{1 - \frac{v^2}{c^2}}}$, where $c$ is the speed of light, 300 million meters per second, and $m$ is the mass of a 100-kilogram object at speed $v$ in m/s. (Example 4) **a–d. See Chapter 11 Answer Appendix.**
    a. Are there any restrictions on the domain of the function? Explain their meaning.
    b. Find $m(10)$, $m(10{,}000)$, and $m(1{,}000{,}000)$.
    c. Describe the behavior of $m(v)$ as $v$ approaches $c$.
    d. Decompose the function into two separate functions.

**30–39. See Chapter 11 Answer Appendix.**
Find two functions $f$ and $g$ such that $h(x) = [f \circ g](x)$. Neither function may be the identity function $f(x) = x$. (Example 4)

30. $h(x) = \sqrt{4x+2} + 7$
31. $h(x) = \frac{6}{x+5} - 8$
32. $h(x) = |4x + 8| - 9$
33. $h(x) = [\![-3(x-9)]\!]$
34. $h(x) = \sqrt{\frac{5-x}{x+2}}$
35. $h(x) = (\sqrt{x} + 4)^3$
36. $h(x) = \frac{6}{(x+2)^2}$
37. $h(x) = \frac{8}{(x-5)^2}$
38. $h(x) = \frac{\sqrt{4+x}}{x-2}$
39. $h(x) = \frac{x+5}{\sqrt{x-1}}$

**40b, d. See Chapter 11 Answer Appendix.**

40. **QUANTUM MECHANICS** The wavelength $\lambda$ of a particle with mass $m$ kg moving at $v$ m/s is represented by $\lambda = \frac{h}{mv}$, where $h$ is a constant equal to $6.626 \cdot 10^{-34}$.
    a. Find a function to represent the wavelength of a 25 kg object as a function of its speed. **$f(v) = \frac{h}{25v}$**
    b. Are there any restrictions on the domain of the function? Explain their meaning.
    c. If the object is traveling 8 m/s, find the wavelength in terms of $h$. **$\lambda = \frac{h}{200}$**
    d. Decompose the function into two separate functions.





## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–42, 87–90, 93, 94, 96–108 | 1–41 odd, 106–108 | 2–42 even, 87–90, 93, 94, 96–105 |
| OL On Level | 1–59 odd, 60, 61–81 odd, 82, 83, 85, 87–90, 93, 94, 96–108 | 1–42, 106–108 | 43–90, 93, 94, 96–105 |
| BL Beyond Level | 43–108 | | |

# 3 Practice

### Formative Assessment

Use Exercises 1–42 to check for understanding.

Then use the table below to customize your assignments for students.

### WatchOut!

**Common Error** In Exercises 15–28, if students evaluate composite functions incorrectly by making the wrong substitutions, emphasize that the second function is the one substituted into the first.

**Common Error** In Exercises 30–39, remind students that there are many solutions. To help them check their answers, ask them to determine the composite and show their work.

### Additional Answers

1. $(f+g)(x) = x^2 + \sqrt{x} + 4$; D = $[0, \infty)$; $(f-g)(x) = x^2 - \sqrt{x} + 4$; D = $[0, \infty)$; $(f \cdot g)(x) = x^{\frac{5}{2}} + 4x^{\frac{1}{2}}$; D = $[0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x^2+4}{\sqrt{x}}$; D = $(0, \infty)$
2. $(f+g)(x) = -x^3 + x + 5$; D = $(-\infty, \infty)$; $(f-g)(x) = -x^3 - x + 11$; D = $(-\infty, \infty)$; $(f \cdot g)(x) = -x^4 + 3x^3 + 8x - 24$; D = $(-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{8-x^3}{x-3}$, D = $(-\infty, 3) \cup (3, \infty)$
3. $(f+g)(x) = x^2 + 6x + 8$; D = $(-\infty, \infty)$; $(f-g)(x) = x^2 + 4x + 4$; D = $(-\infty, \infty)$; $(f \cdot g)(x) = x^3 + 7x^2 + 16x + 12$; D = $(-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = x + 3$; D = $(-\infty, -2) \cup (-2, \infty)$
4. $(f+g)(x) = 2x - 4$; D = $(-\infty, \infty)$; $(f-g)(x) = -14$; D = $(-\infty, \infty)$; $(f \cdot g)(x) = x^2 - 4x - 45$; D = $(-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x-9}{x+5}$; D = $(-\infty, -5) \cup (-5, \infty)$

## Tips for New Teachers

**Formulas** In Exercise 42, remind students that they should use the Pythagorean Theorem for part **a** and the distance formula $d = rt$ for part **b**.

## Additional Answers

**41a.** $h[f(x)]$; The bonus is found after subtracting 300,000 from the total sales.

**42c.** $[d \circ h](t) = \sqrt{75{,}625t^2 + 0.25}$; $d \circ h$ represents the distance of the plane from the control tower as a function of time.

**43–48.** Sample answers given.

**43.** $f(x) = x - \frac{4}{x^2+1}$; $g(x) = \sqrt{x-1}$

**44.** $f(x) = x + \frac{12}{x^2-6}$,
$g(x) = \sqrt{2x+6}$

**45.** $f(x) = \frac{8}{x+2} + 5\sqrt{x}$; $g(x) = x^2$

**46.** $f(x) = \sqrt{x} - \frac{9x}{7}$; $g(x) = -7x$

**47.** $f(x) = \frac{x-4}{2x-9} + \sqrt{\frac{4}{x-4}}$;
$g(x) = x + 4$

**48.** $f(x) = \frac{x^2-4x}{x-2} + \frac{3x-11}{5x-10}$;
$g(x) = x + 2$

**49.** $f(0.5) = -0.75$, $f(-6) = 22$, $f(x+1) = x^2 + 4x + 1$

**50.** $f(0.5) = 8\frac{2}{3}$, $f(-6) = 11\frac{5}{9}$,
$f(x+1) = \frac{2}{x^2+2x+1} + \frac{1}{x+1} - 2x - \frac{7}{3}$

**51.** $f(0.5) = -3.6$, $f(-6) = -300$, $f(x+1) = -9x^2 - 22x + \frac{x+1}{10} - \frac{62}{5}$

**52.** $f(0.5) \approx 2.4$, $f(-6) \approx 650.9$,
$f(x+1) = \sqrt{-x} + 18x^2 + 36x + 18 - \frac{\sqrt{2}}{x+1}$

**60a.** $[I \circ I](x) \approx 1.0323x$, $[I \circ I \circ I](x) \approx 1.0488x$, $[I \circ I \circ I \circ I](x) \approx 1.0656x$

**60b.** The compositions represent the compounded interest for 6 months, 9 months, and 1 year.

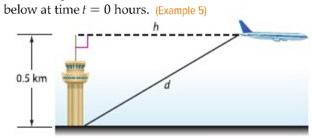
**67a.** $\{m \mid m > 0, m \in \mathbb{R}\}$; The molar mass of the gas cannot be negative or zero.

**41. JOBS** A salesperson for an insurance agency is paid an annual salary plus a bonus of 4% of sales made over AED 300,000. Let $f(x) = x -$ AED 300,000 and $h(x) = 0.04x$, where $x$ is total sales. (Example 5) **a. See margin.**

a. If $x$ is greater than AED 300,000, is the bonus represented by $f[h(x)]$ or by $h[f(x)]$? Explain your reasoning.

b. Determine the amount of bonus for one year with sales of AED 450,000. **AED 6000**

**42. TRAVEL** An airplane flying above a landing strip at 275 km/h passes a control tower 0.5 km below at time $t = 0$ hours. (Example 5)




a. Find the distance $d$ between the airplane and the control tower as a function of the horizontal distance $h$ from the control tower to the plane. $d(h) = \sqrt{h^2 + 0.25}$

b. Find $h$ as a function of time $t$. $h(t) = 275t$

c. Find $d \circ h$. What does this function represent? **See margin.**

d. If the plane continued to fly the same distance from the ground, how far would the plane be from the control tower after 10 minutes? **45.8 km**

**Find two functions $f$ and $g$ such that $h(x) = [f \circ g](x)$. Neither function may be the identity function $f(x) = x$.** **43–48. See margin.**

**43.** $h(x) = \sqrt{x-1} - \frac{4}{x}$

**44.** $h(x) = \sqrt{2x+6} + \frac{6}{x}$

**45.** $h(x) = \frac{8}{x^2+2} + 5|x|$

**46.** $h(x) = \sqrt{-7x} + 9x$

**47.** $h(x) = \frac{x}{2x-1} + \sqrt{\frac{4}{x}}$

**48.** $h(x) = \frac{x^2-4}{x} + \frac{3x-5}{5x}$

**Use the given information to find $f(0.5)$, $f(-6)$, and $f(x+1)$. Round to the nearest tenth if necessary.** **49–52. See margin.**

**49.** $f(x) - g(x) = x^2 + x - 6$, $g(x) = x + 4$

**50.** $f(x) + g(x) = \frac{2}{x^2} + \frac{1}{x} - \frac{1}{3}$, $g(x) = 2x$

**51.** $g(x) - f(x) + \frac{3}{5} = 9x^2 + 4x$, $g(x) = \frac{x}{10}$

**52.** $g(x) = f(x) - 18x^2 + \frac{\sqrt{2}}{x}$, $g(x) = \sqrt{1-x}$

**53–56. See Chapter 11 Answer Appendix.**

**Find $[f \circ g \circ h](x)$.**

**53.** $f(x) = x + 8$
$g(x) = x^2 - 6$
$h(x) = \sqrt{x} + 3$

**54.** $f(x) = x^2 - 2$
$g(x) = 5x + 12$
$h(x) = \sqrt{x} - 4$

**55.** $f(x) = \sqrt{x+5}$
$g(x) = x^2 - 3$
$h(x) = \frac{1}{x}$

**56.** $f(x) = \frac{3}{x}$
$g(x) = x^2 - 4x + 1$
$h(x) = x + 2$

**57.** If $f(x) = x + 2$, find $g(x)$ such that:

a. $(f + g)(x) = x^2 + x + 6$ $g(x) = x^2 + 4$

b. $\left(\frac{f}{g}\right)(x) = \frac{1}{4}$ $g(x) = 4x + 8$

**58.** If $f(x) = \sqrt{4x}$, find $g(x)$ such that:

a. $[f \circ g](x) = |6x|$ $g(x) = 9x^2$

b. $[g \circ f](x) = 200x + 25$ $g(x) = 50x^2 + 25$

**59.** If $f(x) = 4x^2$, find $g(x)$ such that:

a. $(f \cdot g)(x) = x$ $g(x) = \frac{1}{4x}$

b. $(f \cdot g)(x) = \frac{1}{8}x^{\frac{7}{3}}$ $g(x) = \frac{1}{32}x^{\frac{1}{3}}$

**60a–b. See margin.**

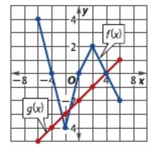
**60. INTEREST** An investment account earns morabaha compounded quarterly. If $x$ dirhams are invested in an account, the investment $I(x)$ after one quarter is the initial investment plus accrued morabaha or $I(x) = 1.016x$.

a. Find $[I \circ I](x)$, $[I \circ I \circ I](x)$, and $[I \circ I \circ I \circ I](x)$.

b. What do the compositions represent?

c. What is the account's annual percentage yield? **about 6.6%**

**Use the graphs of $f(x)$ and $g(x)$ to find each function value.**




**61.** $(f + g)(2)$ **1**

**62.** $(f - g)(-6)$ **9**

**63.** $(f \cdot g)(4)$ **0**

**64.** $\left(\frac{f}{g}\right)(-2)$ $\frac{4}{3}$

**65.** $[f \circ g](-4)$ **0**

**66.** $[g \circ f](6)$ **−3**

**67a, d. See margin.**

**67. CHEMISTRY** The average speed $v(m)$ of gas molecules at 30°C in meters per second can be represented by $v(m) = \sqrt{\frac{(24.9435)(303)}{m}}$, where $m$ is the molar mass of the gas in kilograms per mole.

a. Are there any restrictions on the domain of the function? Explain their meaning.

b. Find the average speed of 145 kilograms per mole gas molecules at 30°C. **about 7.22 m/s**

c. How will the average speed change as the molar mass of gas increases? **The velocity will decrease.**

d. Decompose the function into two separate functions.

**68–71. See margin.**

**Find functions $f$, $g$, and $h$ such that $a(x) = [f \circ g \circ h](x)$.**

**68.** $a(x) = (\sqrt{x-7} + 4)^2$

**69.** $a(x) = \sqrt{(x-5)^2 + 8}$

**70.** $a(x) = \frac{3}{(x-3)^2 + 4}$

**71.** $a(x) = \frac{4}{(\sqrt{x} + 3)^2 + 1}$





## Differentiated Instruction OL BL

**Intrapersonal Learners** Have students use the library or Internet to find applications for using operations with functions and composition of functions. After they identify examples, students should develop their own real-world examples. Each student should develop an example using an operation and an example of a composite function.

**72–77. See Chapter 11 Answer Appendix.**

**For each pair of functions, find $f \circ g$ and $g \circ f$.**

**72.** $f(x) = x^2 - 6x + 5$
$g(x) = \sqrt{x + 4} + 3$

**73.** $f(x) = x^2 + 8x - 3$
$g(x) = \sqrt{x + 19} - 4$

**74.** $f(x) = \sqrt{x + 6}$
$g(x) = \sqrt{16 + x^2}$

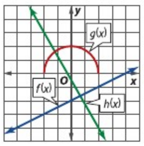
**75.** $f(x) = \sqrt{x}$
$g(x) = \sqrt{9 - x^2}$

**76.** $f(x) = -\frac{8}{5 - 4x}$
$g(x) = \frac{2}{3 + x}$

**77.** $f(x) = \frac{6}{2x + 1}$
$g(x) = \frac{4}{4 - x}$

**78–81. See Chapter 11 Answer Appendix.**

**Graph each of the following.**




**78.** $(f + h)(x)$

**79.** $(h - f)(x)$

**80.** $(f + g)(x)$

**81.** $(h + g)(x)$

**82.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate inverses of functions.

**a.** **ALGEBRAIC** Find the composition of $f$ with $g$ and of $g$ with $f$ for each pair of functions. **For each function, $[f \circ g](x) = [g \circ f](x) = x$.**

| $f(x)$ | $g(x)$ |
|---|---|
| $x + 3$ | $x - 3$ |
| $4x$ | $\frac{x}{4}$ |
| $x^3$ | $\sqrt[3]{x}$ |

**b–e. See Chapter 11 Answer Appendix.**

**b.** **VERBAL** Describe the relationship between the composition of each pair of functions.

**c.** **GRAPHICAL** Graph each pair of functions on the coordinate plane. Graph the line of reflection by finding the midpoint of the segment between corresponding points.

**d.** **VERBAL** Make a conjecture about the line of reflection between the functions.

**e.** **ANALYTICAL** The compositions $[f \circ g](x)$ and $[g \circ f](x)$ are equivalent to which parent function?

**f.** **ANALYTICAL** Find $g(x)$ for each $f(x)$ such that $[f \circ g](x) = [g \circ f](x) = x$.

**i.** $f(x) = x - 6$ **$g(x) = x + 6$**

**ii.** $f(x) = \frac{x}{3}$ **$g(x) = 3x$**

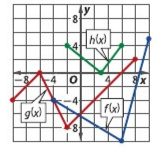
**iii.** $f(x) = x^5$ **$g(x) = \sqrt[5]{x}$**

**iv.** $f(x) = 2x - 3$ **$g(x) = \frac{x + 3}{2}$**

**v.** $f(x) = x^3 + 1$ **$g(x) = \sqrt[3]{x - 1}$**



**State the domain of each composite function.**




**83.** $[f \circ g](x)$ **$\{x \mid -10 \le x \le -4$ or $2 \le x \le 8, x \in \mathbb{R}\}$**

**84.** $[g \circ f](x)$ **$\{x \mid -4 \le x \le 10, x \in \mathbb{R}\}$**

**85.** $[h \circ f](x)$ **$\{x \mid 8 \le x \le 10, x \in \mathbb{R}\}$**

**86.** $[h \circ g](x)$ **$\{x \mid -8 \le x \le -5$ or $4 \le x \le 8, x \in \mathbb{R}\}$**

## H.O.T. Problems Use Higher-Order Thinking Skills

**REASONING** **Determine whether $[f \circ g](x)$ is *even, odd, neither,* or *not enough information* for each of the following.**

**87.** $f$ and $g$ are odd. **odd**

**88.** $f$ and $g$ are even. **even**

**89.** $f$ is even and $g$ is odd. **even**

**90.** $f$ is odd and $g$ is even. **even**

**CHALLENGE** **Find a function $f$ other than $f(x) = x$ such that the following are true.**

**91.** $[f \circ f](x) = x$
**Sample answer: $f(x) = \frac{1}{x}, x \neq 0$**

**92.** $[f \circ f \circ f](x) = f(x)$
**Sample answer: $f(x) = \frac{1}{x} x \neq 0$**

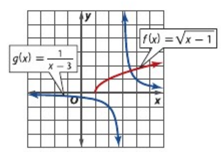
**93.** **WRITING IN MATH** Explain how $f(x)$ might have a domain restriction while $[f \circ g](x)$ might not. Provide an example to justify your reasoning. **See Chapter 11 Answer Appendix.**

**94.** **REASONING** Determine whether the following statement is *true* or *false*. Explain your reasoning.

*If $f$ is a square root function and $g$ is a quadratic function, then $f \circ g$ is always a linear function.* **See Chapter 11 Answer Appendix.**

**95.** **CHALLENGE** State the domain of $[f \circ g \circ h](x)$ for $f(x) = \frac{1}{x - 2}$, $g(x) = \sqrt{x + 1}$, and $h(x) = \frac{4}{x}$.
**$(-\infty, -4] \cup \left(0, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$**

**96.** **WRITING IN MATH** Describe how you would find the domain of $[f \circ g](x)$. **See Chapter 11 Answer Appendix.**




**See Chapter 11 Answer Appendix.**

**97.** **WRITING IN MATH** Explain why order is important when finding the composition of two functions.



## Additional Answers

**67d.** Sample answer: $v(m) = f[g(x)]$;
$f(m) = \sqrt{m}$; $g(m) = \frac{(24.9435)(303)}{m}$

**68–71.** Sample answers given.

**68.** $f(x) = x^2$; $g(x) = \sqrt{x} + 4$; $h(x) = x - 7$

**69.** $f(x) = \sqrt{x}$; $g(x) = x^2 + 8$; $h(x) = x - 5$

**70.** $f(x) = \frac{3}{x}$; $g(x) = x^2 + 4$; $h(x) = x - 3$

**71.** $f(x) = \frac{4}{x}$; $g(x) = x^2 + 1$; $h(x) = \sqrt{x} + 3$

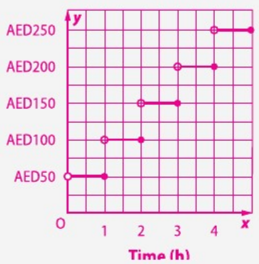
759

## 4 Assess

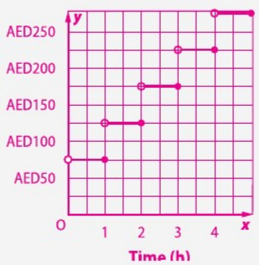
**Yesterday's News** Have students write what they learned in Lesson 11-5 about parent functions and transformations and how the information helped them with operations with functions.

### Additional Answers

**98a.** Cost of Labor




**98b.** Cost of Labor




**105a.**


## Spiral Review

**98. FINANCIAL LITERACY** The cost of labor for servicing cars at B & B Automotive is displayed in the advertisement. (Lesson 11-5)


a. Graph the function that describes the cost for $x$ hours of labor. **a–b. See margin.**

b. Graph the function that would show a AED 25 additional charge if you decide to also get the oil changed and fluids checked.

c. What would be the cost of servicing a car that required 3.45 hours of labor if the owner requested to have the oil changed and the fluids checked? **AED 225**

**Approximate to the nearest hundredth the relative or absolute extrema of each function. State the $x$-values where they occur.** (Lesson 11-4) **99-101. See Chapter 11 Answer Appendix for graphs.**

**99.** $f(x) = 2x^3 - 3x^2 + 4$ **rel. max: (0, 4); rel. min: (1, 3)**

**100.** $g(x) = -x^3 + 5x - 3$ **rel. max: (1.29, 1.30); rel. min: (−1.29, −7.30)**

**101.** $f(x) = x^4 + x^3 - 2$ **abs. min: (−0.75, −2.11)**

**Approximate the real zeros of each function for the given interval.** (Lesson 11-3)

**102.** $g(x) = 2x^5 - 2x^4 - 4x^2 - 1$; $[-1, 3]$ **1.73**

**103.** $f(x) = \frac{x^2 - 3}{x - 4}$; $[-3, 3]$ **−1.73, 1.73**

**104.** $g(x) = \frac{x^2 - 2x - 1}{x^2 + 3x}$; $[1, 5]$ **2.41**

**105. SPORTS** The table shows the leading home run and runs batted in totals in the American League for 2004–2008. (Lesson 11-1)

| Year | 2004 | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|---|
| HR | 43 | 48 | 54 | 54 | 48 |
| RBI | 150 | 148 | 137 | 156 | 146 |

Source: *World Almanac*

a. Make a graph of the data with home runs on the horizontal axis and runs batted in on the vertical axis. **See margin.**

b. Identify the domain and range. **D: {43, 48, 54}, R: {137, 146, 148, 150, 156}**

c. Does the graph represent a function? Explain your reasoning. **No; each of the domain values 48 and 54 is paired with two different range values.**

## Skills Review for Standardized Tests

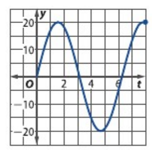
**106. SAT/ACT** A jar contains only red, green, and blue marbles. It is three times as likely that you randomly pick a red marble as a green marble, and five times as likely that you pick a green one as a blue one. Which could be the number of marbles in the jar? **D**

A 39
B 40
C 41
D 42
E 63

**107.** If $g(x) = x^2 + 9x + 21$ and $h(x) = 2(x - 5)^2$, then $h[g(x)] =$ **G**

F $x^4 + 18x^3 + 113x^2 + 288x + 256$
G $2x^4 + 36x^3 + 226x^2 + 576x + 512$
H $3x^4 + 54x^3 + 339x^2 + 864x + 768$
J $4x^4 + 72x^3 + 452x^2 + 1152x + 1024$

**108. FREE RESPONSE** The change in temperature of a substance in degrees Celsius as a function of time for $0 \le t \le 8$ is shown in the graph. **See Chapter 11 Answer Appendix.**




a. This graph represents a function. Explain why.

b. State the domain and range. **D = [0, 8]; R = [−20, 20]**

c. If the initial temperature is 25°C, what is the approximate temperature of the substance at $t = 7$? **about 37.5°C**

d. Analyze the graph for symmetry and zeros. Determine if the function is *even*, *odd*, or *neither*. **See Chapter 11 Answer Appendix.**

e. Is the function continuous at $t = 2$? Explain. **See Chapter 11 Answer Appendix.**

f. Determine the intervals on which the function is increasing or decreasing. **increasing for (0, 1.5) and (4.75, 8); decreasing for (1.5, 4.75)**

g. Estimate the average rate of change for [2, 5]. **about −12.3**

h. What is the significance of your answers to parts **f** and **g** in the context of the situation? **See Chapter 11 Answer Appendix.**





## Differentiated Instruction BL

**Extension** Have each student use one function to find the composition of the function with itself. This is called an iteration. Given a function $f(x)$, and an initial value $x_0$, $f(x_0) = x_1$ is the first iterate, $f[f(x_0)] = f(x_1) = x_2$ is the second iterate, and so on. Have each student find the third iterate of his or her function.

# LESSON 11-7 Inverse Relations and Functions

**Then**

- You found the composition of two functions. (Lesson 11-6)

**Now**

1. Use the horizontal line test to determine inverse functions.
2. Find inverse functions algebraically and graphically.

**Why?**

- The Band Boosters at Hana's high school are selling competition tickets. Table A relates the cost in dirhams to the number of tickets purchased. Table B relates the number of tickets that can be purchased to the number of dirhams spent. By interchanging the input and output from Table A, Hana obtains Table B.

**NewVocabulary**
inverse relation
inverse function
one-to-one

**Table A**

| Tickets | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| Cost (AED) | 2 | 4 | 6 | 8 | 10 |

**Table B**

| Money Spent (AED) | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Tickets | 1 | 2 | 3 | 4 | 6 |

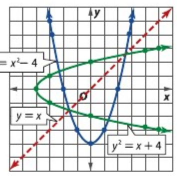
## 1 Inverse Functions

The relation shown in Table A is the *inverse relation* of the relation shown in Table B. **Inverse relations** exist if and only if one relation contains $(b, a)$ whenever the other relation contains $(a, b)$. When a relation is expressed as an equation, its inverse relation can be found by interchanging the independent and dependent variables. Consider the following.

**Relation**

$y = x^2 - 4$

| $x$ | $y$ |
|---|---|
| −3 | 5 |
| −2 | 0 |
| −1 | −3 |
| 0 | −4 |
| 1 | −3 |
| 2 | 0 |
| 3 | 5 |




**Inverse Relation**

$x = y^2 - 4$ or $y^2 = x + 4$

| $x$ | $y$ |
|---|---|
| 5 | −3 |
| 0 | −2 |
| −3 | −1 |
| −4 | 0 |
| −3 | 1 |
| 0 | 2 |
| 5 | 3 |

Notice that these inverse relations are reflections of each other in the line $y = x$. This relationship is true for the graphs of all relations and their inverse relations. We are most interested in *functions* with inverse relations that are also *functions*. If the inverse relation of a function $f$ is also a function, then it is called the **inverse function** of $f$ and is denoted $f^{-1}$, read *f inverse*.

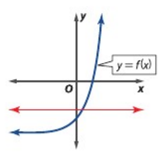
Not all functions have inverse functions. In the graph above, notice that the original relation is a function because it passes the vertical line test. But its inverse relation fails this test, so it is not a function. The reflective relationship between the graph of a function and its inverse relation leads us to the following graphical test for determining whether the inverse function of a function exists.

### KeyConcept Horizontal Line Test

**Words** A function $f$ has an inverse function $f^{-1}$ if and only if each horizontal line intersects the graph of the function in at most one point.

**Example** Since no horizontal line intersects the graph of $f$ more than once, the inverse function $f^{-1}$ exists.

**Model**








## 1 Focus

**VerticalAlignment**

**Before Lesson 11-7** Find the composition of two functions.

**Lesson 11-7** Use the graphs of functions to determine if they have inverse functions.
Find inverse functions algebraically and graphically.

**After Lesson 11-7** Analyze graphs of polynomial and rational functions.

## 2 Teach

### Scaffolding Questions

Have students read the **Why?** section of the lesson. Have students think about relations and their inverses.

**Ask:**

- What is a function for the area of a square? $A(s) = s^2$
- What is the area of a square when the side of the square measures 5? 25

(*continued on the next page*)

**WatchOut!**

**Horizontal Line Test** When using a graphing calculator, closely examine places where it appears that the function may fail the horizontal line test. Use Zoom In and Zoom Out features, or adjust the window to be sure.

**Example 1** Apply the Horizontal Line Test

**Graph each function using a graphing calculator, and apply the horizontal line test to determine whether its inverse function exists. Write *yes* or *no*.**

**a.** $f(x) = |x - 1|$

The graph of $f(x)$ in Figure 11.7.1 shows that it is possible to find a horizontal line that intersects the graph of $f(x)$ more than once. Therefore, you can conclude that $f^{-1}$ does not exist.

**b.** $g(x) = x^3 - 6x^2 + 12x - 8$

The graph of $g(x)$ in Figure 11.7.2 shows that it is not possible to find a horizontal line that intersects the graph of $g(x)$ in more than one point. Therefore, you can conclude that $g^{-1}$ exists.

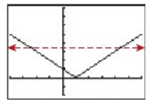


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**Figure 11.7.1**

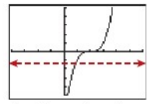

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**Figure 11.7.2**

**GuidedPractice**

**1A.** $h(x) = \frac{4}{x}$ yes

**1B.** $f(x) = x^2 + 5x - 7$ no

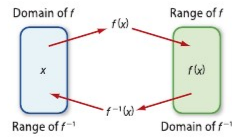
**ReadingMath**

**Inverse Function Notation** The symbol $f^{-1}(x)$ should not be confused with the reciprocal function $\frac{1}{f(x)}$. If $f$ is a function, the symbol $f^{-1}$ can only be interpreted as $f$ inverse of $x$.

## 2 Find Inverse Functions

If a function passes the horizontal line test, then it is said to be **one-to-one**, because no $x$-value is matched with more than one $y$-value and no $y$-value is matched with more than one $x$-value.

If a function $f$ is one-to-one, it has an inverse function $f^{-1}$ such that the domain of $f$ is equal to the range of $f^{-1}$, and the range of $f$ is equal to the domain of $f^{-1}$.




To find an inverse function algebraically, follow the steps below.

**KeyConcept** Finding an Inverse Function

**Step 1** Determine whether the function has an inverse by checking to see if it is one-to-one using the horizontal line test.

**Step 2** In the equation for $f(x)$, replace $f(x)$ with $y$ and then interchange $x$ and $y$.

**Step 3** Solve for $y$ and then replace $y$ with $f^{-1}(x)$ in the new equation.

**Step 4** State any restrictions on the domain of $f^{-1}$. Then show that the domain of $f$ is equal to the range of $f^{-1}$ and the range of $f$ is equal the domain of $f^{-1}$.

The last step implies that only part of the function you find algebraically may be the inverse function of $f$. Therefore, be sure to analyze the domain of $f$ when finding $f^{-1}$.





- Write a function that represents the side of a square given the area. What is the length of the side of a square if the area of the square is 100? $S = \sqrt{A}$; 10
- Write a function for distance if rate is constant and time is variable. Write a function for time if distance is variable and rate is a constant. $d = f(t) = rt$; $t = f(d) = \frac{d}{r}$

## 1 Inverse Functions

**Example 1** shows how to determine graphically if the inverse function of a function exists.

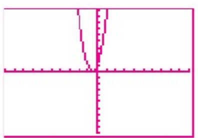
### Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

**Additional Example**

**1** Graph each function using a graphing calculator, and apply the horizontal line test to determine whether its inverse function exists. Write *yes* or *no*.

**a.** $y = 4x^2 + 4x + 1$



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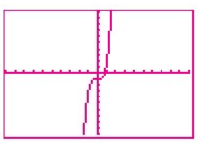
**b.** $f(x) = x^5 + x^3 - 1$



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**ReadingMath**

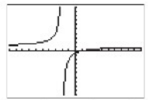
**Invertible Functions** A function that has an inverse function is said to be *invertible*.

## Example 2 Find Inverse Functions Algebraically

**Determine whether $f$ has an inverse function. If it does, find the inverse function and state any restrictions on its domain.**

**a.** $f(x) = \frac{x-1}{x+2}$

The graph of $f$ shown passes the horizontal line test. Therefore, $f$ is a one-to-one function and has an inverse function. From the graph, you can see that $f$ has domain $(-\infty, -2) \cup (-2, \infty)$ and range $(-\infty, 1) \cup (1, \infty)$. Now find $f^{-1}$.



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$f(x) = \frac{x-1}{x+2}$ Original function

$y = \frac{x-1}{x+2}$ Replace $f(x)$ with $y$.

$x = \frac{y-1}{y+2}$ Interchange $x$ and $y$.

$xy + 2x = y - 1$ Multiply each side by $y + 2$. Then apply the Distributive Property.

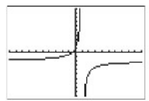
$xy - y = -2x - 1$ Isolate the $y$-terms.

$y(x - 1) = -2x - 1$ Distributive Property

$y = \frac{-2x-1}{x-1}$ Solve for $y$.

$f^{-1}(x) = \frac{-2x-1}{x-1}$ Replace $y$ with $f^{-1}(x)$. Note that $x \neq 1$.

From the graph at the right, you can see that $f^{-1}$ has domain $(-\infty, 1) \cup (1, \infty)$ and range $(-\infty, -2) \cup (-2, \infty)$. The domain and range of $f$ are equal to the range and domain of $f^{-1}$, respectively. So, $f^{-1}(x) = \frac{-2x-1}{x-1}$ for $x \neq 1$.



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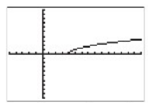
**b.** $f(x) = \sqrt{x-4}$

The graph of $f$ shown passes the horizontal line test. Therefore, $f$ is a one-to-one function and has an inverse function. From the graph, you can see that $f$ has domain $[4, \infty)$ and range $[0, \infty)$. Now find $f^{-1}$.



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$f(x) = \sqrt{x-4}$ Original function

$y = \sqrt{x-4}$ Replace $f(x)$ with $y$.

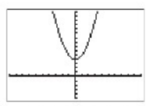
$x = \sqrt{y-4}$ Interchange $x$ and $y$.

$x^2 = y - 4$ Square each side.

$y = x^2 + 4$ Solve for $y$.

$f^{-1}(x) = x^2 + 4$ Replace $y$ with $f^{-1}(x)$.

From the graph of $y = x^2 + 4$ shown, you can see that the inverse relation has domain $(-\infty, \infty)$ and range $[4, \infty)$. By restricting the domain of the inverse relation to $[0, \infty)$, the domain and range of $f$ are equal to the range and domain of $f^{-1}$, respectively. So, $f^{-1}(x) = x^2 + 4$, for $x \geq 0$.



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**Guided Practice**

**2A.** $f(x) = -16 + x^3$ yes; $f^{-1}(x) = \sqrt[3]{x+16}$

**2B.** $f(x) = \frac{x+7}{x}$ yes; $f^{-1}(x) = -\frac{7}{1-x}, x \neq 1$

**2C.** $f(x) = \sqrt{x^2 - 20}$ no





## 2 Find Inverse Functions

**Example 2** shows how to find the inverse of a function algebraically. **Example 3** shows how to verify that functions are inverses. **Example 4** shows how to find an inverse function graphically. **Example 5** shows how to use inverse functions.

### Additional Example

**2** Determine whether $f$ has an inverse function. If it does, find the inverse function and state any restrictions on its domain.

**a.** $f(x) = \frac{x}{2x-1}$ $f^{-1}$ exists; $f^{-1}(x) = \frac{x}{2x-1}, x \neq \frac{1}{2}$

**b.** $f(x) = 2\sqrt{x-1}$ $f^{-1}$ exists with domain $[0, \infty)$; $f^{-1}(x) = \frac{x^2}{4} + 1$

### Teach with Tech

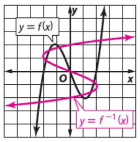
**Graphing Calculator** Have students work in pairs, with each pair using one graphing calculator. One partner identifies a function. The second partner graphs the function. If the function passes the horizontal line test, the first partner then algebraically determines the inverse function. The second partner graphs the inverse to be sure the inverse and the function are symmetric in the line $y = x$. Have each pair take turns with each role. Pairs should find at least four functions that have an inverse function.

## Additional Examples

**3** Show that $f(x) = \frac{2}{3}x + 2$ and $g(x) = \frac{3}{2}(x - 2)$ are inverse functions.

$$f[g(x)] = f\left(\frac{3}{2}(x-2)\right)$$
$$= \frac{2}{3}\left(\frac{3}{2}(x-2) + 2\right)$$
$$= x - 2 + 2$$
$$= x$$
$$g[f(x)] = g\left(\frac{2}{3}x + 2\right)$$
$$= \frac{3}{2}\left(\frac{2}{3}x + 2 - 2\right)$$
$$= \frac{3}{2}\left(\frac{2}{3}x\right)$$
$$= x$$

**4** Use the graph of $f(x)$ to sketch the graph of $f^{-1}(x)$.




### Focus on Mathematical Content

**Horizontal Line Test** The graph of the inverse of a function is the reflection of the original function in the line $y = x$. Since the vertical line test shows whether a relation is a function, it can also be reflected in the line $y = x$. This results in a horizontal line which can be used with the graph of the inverse.

An inverse function $f^{-1}$ has the effect of "undoing" the action of a function $f$. For this reason, inverse functions can also be defined in terms of their composition with each other.

**StudyTip**

**Inverse Functions** The biconditional statement "if and only if" in the definition of inverse functions means that if $g$ is the inverse of $f$, then it is also true that $f$ is the inverse of $g$.

**KeyConcept** Compositions of Inverse Functions

Two functions, $f$ and $g$, are inverse functions if and only if
- $f[g(x)] = x$ for every $x$ in the domain of $g(x)$ and
- $g[f(x)] = x$ for every $x$ in the domain of $f(x)$.

Notice that the composition of a function with its inverse function is always the identity function. You can use this fact to verify that two functions are inverse functions of each other.

### Example 3 Verify Inverse Functions

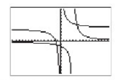
**Show that $f(x) = \frac{6}{x-4}$ and $g(x) = \frac{6}{x} + 4$ are inverse functions.**

Show that $f[g(x)] = x$ and that $g[f(x)] = x$.

$$f[g(x)] = f\left(\frac{6}{x} + 4\right) = \frac{6}{\left(\frac{6}{x} + 4\right) - 4} = \frac{6}{\left(\frac{6}{x}\right)} \text{ or } x$$

$$g[f(x)] = g\left(\frac{6}{x-4} + 4\right) = \frac{6}{\left(\frac{6}{x-4}\right)} + 4 = x - 4 + 4 \text{ or } x$$

Because $f[g(x)] = g[f(x)] = x$, $f(x)$ and $g(x)$ are inverse functions. This is supported graphically because $f(x)$ and $g(x)$ appear to be reflections of each other in the line $y = x$.



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**GuidedPractice**

**Show that $f$ and $g$ are inverse functions.**

**3A.** $f(x) = 18 - 3x$, $g(x) = 6 - \frac{x}{3}$

**3B.** $f(x) = x^2 + 10$, $x \geq 0$; $g(x) = \sqrt{x - 10}$

**3A.**
$$f[g(x)] = 18 - 3\left(6 - \frac{x}{3}\right)$$
$$= 18 - 18 + x$$
$$= x;$$
$$g[f(x)] = 6 - \frac{18 - 3x}{3}$$
$$= 6 - 6 + x$$
$$= x$$

**3B.**
$$f[g(x)] = \left(\sqrt{x - 10}\right)^2 + 10$$
$$= x - 10 + 10$$
$$= x;$$
$$g[f(x)] = \sqrt{x^2 + 10 - 10}$$
$$= x$$

The inverse functions of most one-to-one functions are often difficult to find algebraically. However, it is possible to graph the inverse function by reflecting the graph of the original function in the line $y = x$.

**StudyTip**

**Inverse Functions and Extrema** A continuous function has an inverse function if and only if it has no local maxima or minima. If the function does have a local maximum or minimum, then it will not pass the horizontal line test, and is not a one-to-one function.

### Example 4 Find Inverse Functions Graphically

**Use the graph of $f(x)$ in Figure 11.7.3 to graph $f^{-1}(x)$.**

Graph the line $y = x$. Locate a few points on the graph of $f(x)$. Reflect these points in $y = x$. Then connect them with a smooth curve that mirrors the curvature of $f(x)$ in line $y = x$ (Figure 11.7.4).

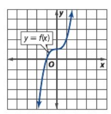



Figure 11.7.3

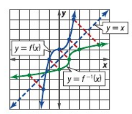


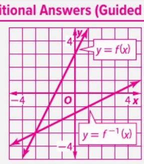

Figure 11.7.4



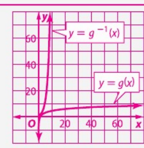


## Additional Answers (Guided Practice)

**4A.**



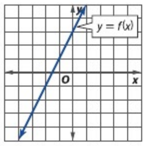

**4B.**



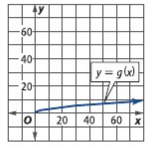
## Guided Practice

Use the graph of each function to graph its inverse function. **4A–B. See margin.**

4A.

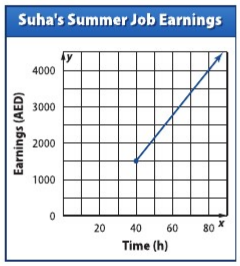



4B.




## Real-World Example 5 Use an Inverse Function

**SUMMER EARNINGS** **Suha earns AED 40 an hour, works at least 40 hours per week, and receives overtime pay at 1.5 times her regular hourly rate for any time over 40 hours. Her total earnings $f(x)$ for a week in which she worked $x$ hours is given by $f(x) = 1600 + 60(x - 40)$.**




**a. Explain why the inverse function $f^{-1}(x)$ exists. Then find $f^{-1}(x)$.**

The function simplifies to $f(x) = 1600 + 60x - 2400$ or $60x - 800$. The graph of $f(x)$ passes the horizontal line test. Therefore, $f(x)$ is a one-to-one function and has an inverse function.

Find $f^{-1}(x)$.

$$f(x) = 60x - 800 \quad \text{Original function}$$
$$y = 60x - 800 \quad \text{Replace } f(x) \text{ with } y.$$
$$x = 60y - 800 \quad \text{Interchange } x \text{ and } y.$$
$$x + 800 = 60y \quad \text{Add 800 to each side.}$$
$$y = \frac{x + 800}{60} \quad \text{Solve for } y.$$
$$f^{-1}(x) = \frac{x + 800}{60} \quad \text{Replace } y \text{ with } f^{-1}(x).$$

**b. What do $f^{-1}(x)$ and $x$ represent in the inverse function?**

In the inverse function, $x$ represents Suha's earnings for a particular week and $f^{-1}(x)$ represents the number of hours Suha worked that week.

**c. What restrictions, if any, should be placed on the domain of $f(x)$ and $f^{-1}(x)$? Explain.**

The function $f(x)$ assumes that Suha works at least 40 hours in a week. There are $7 \cdot 24$ or 168 hours in a week, so the domain of $f(x)$ is [40, 168]. Because $f(40) = 1600$ and $f(168) = 9280$, the range of $f(x)$ is [1600, 9280]. Because the range of $f(x)$ must equal the domain of $f^{-1}(x)$, the domain of $f^{-1}(x)$ is [1600, 9280].

**d. Find the number of hours Suha worked last week if her earnings were AED 1900.**

Because $f^{-1}(1900) = \frac{1900 + 800}{60}$ or 45, Suha worked 45 hours last week.

## Guided Practice

5. **SAVINGS** Amal's net pay is 65% of her gross pay, and she budgets AED 2400 per month for living expenses. She estimates that she can save 20% of her remaining money, so her one-month savings $f(x)$ for a gross pay of $x$ dirhams is given by $f(x) = 0.2(0.65x - 2400)$.
   - **A.** Explain why the inverse function $f^{-1}(x)$ exists. Then find $f^{-1}(x)$.
   - **B.** What do $f^{-1}(x)$ and $x$ represent in the inverse function?
   - **C.** What restrictions, if any, should be placed on the domains of $f(x)$ and $f^{-1}(x)$? Explain.
   - **D.** Determine Amal's gross pay for one month if her savings for that month were AED 480.

**5A.** $f^{-1}$ exists because $f$ passes the horizontal line test and is therefore one-to-one; $f^{-1}(x) = \frac{100x}{13} + \frac{48{,}000}{13}$.

**5B.** $f^{-1}(x)$ represents Amal's gross pay for a month, and $x$ represents her savings per month.

**5C.** $x \geq 3692.32$

**5D.** AED 7384.60





## Additional Example

**5 MANUFACTURING** The fixed costs for manufacturing one type of stereo system are AED 96,000 with variable cost of AED 80 per unit. The total cost $f(x)$ of making $x$ stereos is given by $f(x) = 96{,}000 + 80x$.

**a.** Explain why the inverse function $f^{-1}(x)$ exists. Then find $f^{-1}(x)$. The graph of $f(x)$ passes the horizontal line test. $f^{-1}(x) = \frac{x - 96{,}000}{80}$

**b.** What do $f^{-1}(x)$ and $x$ represent in the inverse function? In the inverse function, $x$ represents the total cost and $f^{-1}(x)$ represents the number of stereos.

**c.** What restrictions, if any, should be placed on the domain of $f(x)$ and $f^{-1}(x)$? Explain. The domain of $f(x)$ has to be nonnegative integers. The domain of $f^{-1}(x)$ is multiples of 80 greater than 96,000.

**d.** Find the number of stereos made if the total cost was AED 216,000. 1500 stereos

## Follow-up

Students have explored functions and inverse functions.

**Ask:**

- How can the inverse of a function be used to help interpret a real-world event or solve a problem? Sample answer: Inverse relationships "undo" each other. Having an inverse of a function allows you to model a relationship using either quantity as the independent variable.

## Differentiated Instruction AL OL

**Kinesthetic Learners** On a large coordinate grid, students should graph the identity function $f(x) = x$ using a noticeable color, length of string, or something similar. Have them graph points on the function $f(x) = x^3$ for $x$-values −3, −2, −1, 0, 1, 2, and 3. Then have them graph the inverse function by reflecting those points in the line $y = x$. Have them make a table with coordinate pairs for both functions. Then they use their tables to explain why the first step for calculating an inverse function algebraically is to switch $y$ and $x$ in the original equation.

# 3 Practice

## Formative Assessment

Use Exercises 1–45 to check for understanding.

Then use the table below to customize your assignments for students.

**WatchOut!**

**Common Error** Students may mistakenly try to find $f^{-1}(x)$ by finding $\frac{1}{f(x)}$. Remind them that $f^{-1}$ is a symbol and not a variable to the $-1$ power. In other words, $f^{-1}$ is the inverse of $f$, while $\frac{1}{f}$ is the reciprocal of $f$.

**Common Error** In Exercises 27–36, help students work through the substitution and simplification by reminding them to use parentheses correctly when substituting.

## Additional Answers

**16.** yes; $f^{-1}(x) = x^2 - 8; x \geq 0$

**19.** yes; $f^{-1}(x) = \frac{4}{x+1}; x \neq -1$

**20.** yes; $g^{-1}(x) = \frac{-6}{x-1}; x \neq 1$

**21.** yes; $f^{-1}(x) = 8 - \frac{36}{x^2}; x > 0$

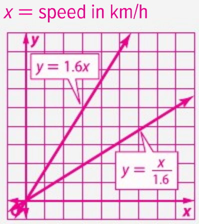
**22.** yes; $g^{-1}(x) = -3 + \frac{49}{x^2}; x > 0$

**23.** yes; $f^{-1}(x) = \frac{8x+3}{x-6}; x \neq 6$

**24.** yes; $h^{-1}(x) = \frac{5x+4}{3x-1}; x \neq \frac{1}{3}$

**26a.** $y = \frac{x}{1.6}$; $y$ = speed in km/h, $x$ = speed in km/h

**26b.**




## Exercises

**Graph each function using a graphing calculator, and apply the horizontal line test to determine whether its inverse function exists. Write *yes* or *no*.** (Example 1)

1. $f(x) = x^2 + 6x + 9$ **no**
2. $f(x) = x^2 - 16x + 64$ **no**
3. $f(x) = x^2 - 10x + 25$ **no**
4. $f(x) = 3x - 8$ **yes**
5. $f(x) = \sqrt{2x}$ **yes**
6. $f(x) = 4$ **no**
7. $f(x) = \sqrt{x+4}$ **yes**
8. $f(x) = -4x^2 + 8$ **no**
9. $f(x) = \frac{5}{x-6}$ **yes**
10. $f(x) = \frac{8}{x+2}$ **yes**
11. $f(x) = x^3 - 9$ **yes**
12. $f(x) = \frac{1}{4}x^3$ **yes**

**Determine whether each function has an inverse function. If it does, find the inverse function and state any restrictions on its domain.** (Example 2) **16, 19–24. See margin.**

13. $g(x) = -3x^4 + 6x^2 - x$ **no**
14. $f(x) = 4x^5 - 8x^4$ **no**
15. $h(x) = x^7 + 2x^3 - 10x^2$ **no**
16. $f(x) = \sqrt{x+8}$
17. $f(x) = \sqrt{6 - x^2}$ **no**
18. $f(x) = |x - 6|$ **no**
19. $f(x) = \frac{4-x}{x}$
20. $g(x) = \frac{x-6}{x}$
21. $f(x) = \frac{6}{\sqrt{8-x}}$
22. $g(x) = \frac{7}{\sqrt{x+3}}$
23. $f(x) = \frac{6x+3}{x-8}$
24. $h(x) = \frac{x+4}{3x-5}$
25. $g(x) = |x+1| + |x-4|$ **no**

26. **SPEED** The speed of an object in kilometers per hour $y$ is $y = 1.6x$, where $x$ is the speed of the object in km/h. (Example 2) **a–b. See margin.**
    a. Find an equation for the inverse of the function. What does each variable represent?
    b. Graph each equation on the same coordinate plane.

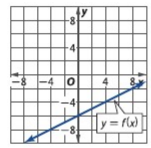
**Show algebraically that $f$ and $g$ are inverse functions.** (Example 3) **27–36. See Chapter 11 Answer Appendix.**

27. $f(x) = -6x + 3$; $g(x) = \frac{3-x}{6}$
28. $f(x) = 4x + 9$; $g(x) = \frac{x-9}{4}$
29. $f(x) = -3x^2 + 5, x \geq 0$; $g(x) = \sqrt{\frac{5-x}{3}}$
30. $f(x) = \frac{x^2}{4} + 8, x \geq 0$; $g(x) = \sqrt{4x - 32}$
31. $f(x) = 2x^3 - 6$; $g(x) = \sqrt[3]{\frac{x+6}{2}}$
32. $f(x) = (x+8)^{\frac{3}{2}}$; $g(x) = x^{\frac{2}{3}} - 8, x \geq 0$
33. $g(x) = \sqrt{x+8} - 4$; $f(x) = x^2 + 8x + 8, x \geq -4$
34. $g(x) = \sqrt{x-8} + 5$; $f(x) = x^2 - 10x + 33, x \geq 5$
35. $f(x) = \frac{x+4}{x}$; $g(x) = \frac{4}{x-1}$
36. $f(x) = \frac{x-6}{x+2}$; $g(x) = \frac{2x+6}{1-x}$

37. **PHYSICS** The kinetic energy of an object in motion in joules can be described by $f(x) = 0.5mx^2$, where $m$ is the mass of the object in kilograms and $x$ is the speed of the object in meters per second. (Example 3) **a–c. See Chapter 11 Answer Appendix.**
    a. Find the inverse of the function. What does each variable represent?
    b. Show that $f(x)$ and the function you found in part **a** are inverses.
    c. Graph $f(x)$ and $f^{-1}(x)$ on the same graphing calculator screen if the mass of the object is 1 kg.

**38–43. See Chapter 11 Answer Appendix.**
**Use the graph of each function to graph its inverse function.** (Example 4)

38.

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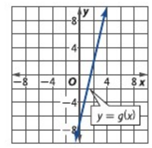


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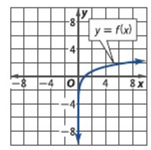


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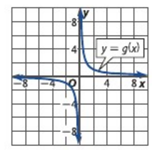


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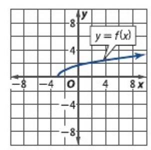
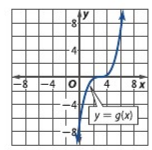



43.




44. **JOBS** Suha sells shoes at a department store after school. Her base salary each week is AED 560, and she earns a 10% commission on each pair of shoes that she sells. Her total earnings $f(x)$ for a week in which she sold $x$ dirhams worth of shoes is $f(x) = 560 + 0.1x$. (Example 5) **a–c. See Chapter 11 Answer Appendix.**
    a. Explain why the inverse function $f^{-1}(x)$ exists. Then find $f^{-1}(x)$.
    b. What do $f^{-1}(x)$ and $x$ represent in the inverse function?
    c. What restrictions, if any, should be placed on the domains of $f(x)$ and $f^{-1}(x)$? Explain.
    d. Find Suha's total sales last week if her earnings for that week were AED 880. **AED 3200**





## Differentiated Homework Options OL BL AL

| Level | Assignment | Two-Day Option | |
|---|---|---|---|
| AL Approaching Level | 1–45, 83–85, 87–104 | 1–45 odd, 101–104 | 2–42 even, 83–85, 87–100 |
| OL On Level | 1–53 odd, 54, 55–63 odd, 64, 65, 67, 68, 69–79 odd, 81–85, 87–104 | 1–45, 101–104 | 46–85, 87–100 |
| BL Beyond Level | 46–104 | | |

**45.** **CURRENCY** The average exchange rate from Euros to U.S. dollars for a recent four-month period can be described by $f(x) = 0.66x$, where $x$ is the currency value in Euros. (Example 5)

a. Explain why the inverse function $f^{-1}(x)$ exists. Then find $f^{-1}(x)$. **a–b. See margin.**

b. What do $f^{-1}(x)$ and $x$ represent in the inverse function?

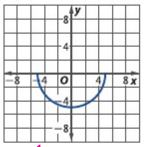
c. What restrictions, if any, should be placed on the domains of $f(x)$ and $f^{-1}(x)$? Explain.

d. What is the value in Euros of 100 U.S. dollars? **151.52**

**c. $x \geq 0$; You cannot exchange negative money.**

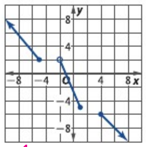
**Determine whether each function has an inverse function.**

**46.**



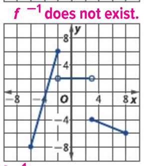
**$f^{-1}$ does not exist.**

**47.**



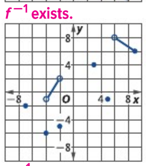
**$f^{-1}$ exists.**

**48.**



**$f^{-1}$ does not exist.**

**49.**



**$f^{-1}$ exists.**

**Determine if $f^{-1}$ exists. If so, complete a table for $f^{-1}$.** **50–53. See margin.**

**50.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −6 | −4 | −1 | 3 | 6 | 10 |
| $f(x)$ | −4 | 0 | 3 | 5 | 9 | 13 |

**51.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −3 | −2 | −1 | 0 | 1 | 2 |
| $f(x)$ | 14 | 11 | 8 | 10 | 11 | 16 |

**52.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $f(x)$ | 2 | 8 | 16 | 54 | 27 | 16 |

**53.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −10 | −9 | −8 | −7 | −6 | −5 |
| $f(x)$ | 8 | 7 | 6 | 5 | 4 | 3 |

**54.** **TEMPERATURE** The formula $f(x) = \frac{9}{5}x + 32$ is used to convert $x$ degrees Celsius to degrees Fahrenheit. To convert $x$ degrees Fahrenheit to Kelvin, the formula $k(x) = \frac{5}{9}(x + 459.67)$ is used. **a–d. See margin.**

a. Find $f^{-1}$. What does this function represent?

b. Show that $f$ and $f^{-1}$ are inverse functions. Graph each function on the same graphing calculator screen.

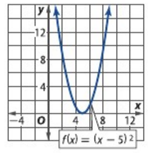
c. Find $[k \circ f](x)$. What does this function represent?

d. If the temperature is 60°C, what would the temperature be in Kelvin?

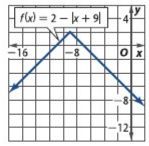


**Restrict the domain of each function so that the resulting function is one-to-one. Then determine the inverse of the function.** **55–58. See margin.**

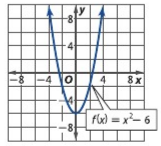
**55.**



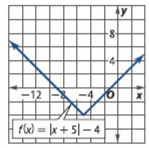

**56.**




**57.**




**58.**




**59–62. See margin.**

**State the domain and range of $f$ and $f^{-1}$, if $f^{-1}$ exists.**

**59.** $f(x) = \sqrt{x - 6}$

**60.** $f(x) = x^2 + 9$

**61.** $f(x) = \frac{3x + 1}{x - 4}$

**62.** $f(x) = \frac{8x + 3}{2x - 6}$

**63.** **ENVIRONMENT** Once an endangered species, the bald eagle was downlisted to threatened status in 1995. The table shows the number of nesting pairs each year.

| Year | Nesting Pairs |
|---|---|
| 1984 | 1757 |
| 1990 | 3035 |
| 1994 | 4449 |
| 1998 | 5748 |
| 2000 | 6471 |
| 2005 | 7066 |

**a–b. See Chapter 11 Answer Appendix.**

a. Use the table to approximate a linear function that relates the number of nesting pairs to the year. Let 0 represent 1984.

b. Find the inverse of the function you generated in part **a.** What does each variable represent?

c. Using the inverse function, in approximately what year was the number of nesting pairs 5094? **1997**

**64.** **FLOWERS** Nisreen needs to purchase 75 flower stems for banquet decorations. She can choose between lilies and hydrangea, which cost AED 5.00 per stem and AED 3.50 per stem, respectively. **a–c. See Chapter 11 Answer Appendix.**

a. Write a function for the total cost of the flowers.

b. Find the inverse of the cost function. What does each variable represent?

c. Find the domain of the cost function and its inverse.

d. If the total cost for the flowers was AED 307.50, how many lilies did Nisreen purchase? **30**



**WatchOut!**

**Common Error** In Exercises 50–54, students may have difficulty because there is no graph. Have these students graph $f(x)$ and apply the horizontal line test.

## Additional Answers

**45a.** Sample answer: The graph of the function is linear, so it passes the horizontal line test. Therefore, it is a one-to-one function and it has an inverse; $f^{-1}(x) = \frac{x}{0.66}$.

**45b.** $x$ represents the value of the currency in U.S. dollars, and $f^{-1}(x)$ represents the value of the currency in Euros.

**50.** $f^{-1}$ exists.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −4 | 0 | 3 | 5 | 9 | 13 |
| $f^{-1}(x)$ | −6 | −4 | −1 | 3 | 6 | 10 |

**51.** $f^{-1}$ does not exist.

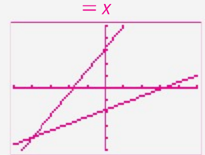
**52.** $f^{-1}$ does not exist.

**53.** $f^{-1}$ exists.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 8 | 7 | 6 | 5 | 4 | 3 |
| $f^{-1}(x)$ | −10 | −9 | −8 | −7 | −6 | −5 |

**54a.** $f^{-1}(x) = \frac{5}{9}(x - 32)$; $f^{-1}$ represents the formula used to convert degrees Fahrenheit to degrees Celsius.

**54b.**
$$f[f^{-1}(x)] = \frac{9}{5}\left[\frac{5}{9}(x - 32) + 32\right]$$
$$= \frac{9}{5}\left(\frac{5}{9}x - \frac{160}{9}\right) + 32$$
$$= x - 32 + 32$$
$$= x$$
$$f^{-1}[f(x)] = \frac{5}{9}\left[\left(\frac{9}{5}x + 32\right) - 32\right]$$
$$= \frac{5}{9}\left(\frac{9}{5}x\right)$$
$$= x$$



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**54c.** $k[f(x)] = x + 273.15$; represents the formula used to convert degrees Celsius to degrees Kelvin.

**54d.** 333.15 K

**55.** Sample answer: $x \geq 5$; $f^{-1}(x) = \sqrt{x} + 5$

**56.** Sample answer: $x \leq -9$; $f^{-1}(x) = x - 11$, $x \leq 2$

**57.** Sample answer: $x \geq 0$, $f^{-1}(x) = \sqrt{x + 6}$

**58.** Sample answer: $x \geq -5$; $f^{-1}(x) = x - 1$, $x \geq -4$

**59.** $f$: D = $\{x \mid x \geq 6, x \in \mathbb{R}\}$, R = $\{y \mid y \geq 0, y \in \mathbb{R}\}$
$f^{-1}$: D = $\{x \mid x \geq 0, y \in \mathbb{R}\}$, R = $\{y \mid y \geq 6, x \in \mathbb{R}\}$

**60.** $f^{-1}$ does not exist.

**61.** $f$: D = $\{x \mid x \neq 4, x \in \mathbb{R}\}$, R = $\{y \mid y \neq 3, x \in \mathbb{R}\}$
$f^{-1}$: D = $\{x \mid x \neq 3, x \in \mathbb{R}\}$, R = $\{y \mid y \neq 4, x \in \mathbb{R}\}$

**62.** $f$: D = $\{x \mid x \neq 3, x \in \mathbb{R}\}$, R = $\{y \mid y \neq 4, x \in \mathbb{R}\}$
$f^{-1}$: D = $\{x \mid x \neq 4, x \in \mathbb{R}\}$, R = $\{y \mid y \neq 3, x \in \mathbb{R}\}$

## Additional Answer

**68d.** $T^{-1}(x) = \frac{x + 50}{0.9}$; the inverse represents the original price of the phone as a function of the price of the phone after the rebate and the discount.

**75.** $\sqrt{x^2 + 3}$ for $x \geq 0$

**77.** $x + 3$ for $x \geq 1$

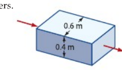
**79.** $x^4 + 5x^2 + 4$ for $x \geq 0$

**Find an equation for the inverse of each function, if it exists. Then graph the equations on the same coordinate plane. Include any domain restrictions.**

**65.** $f(x) = \begin{cases} x^2 & \text{if } -4 \geq x \\ -2x + 5 & \text{if } -4 < x \end{cases}$ **See Chapter 11 Answer Appendix.**

**66.** $f(x) = \begin{cases} -4x + 6 & \text{if } -5 \geq x \\ 2x - 8 & \text{if } -5 < x \end{cases}$ **$f^{-1}(x)$ does not exist.**

**67.** **FLOW RATE** The flow rate of a gas is the volume of gas that passes through an area during a given period of time. The speed $v$ of air flowing through a vent can be found using $v(r) = \frac{r}{A}$, where $r$ is the flow rate in cubic meters per second and $A$ is the cross-sectional area of the vent in square meters.




**a. $v^{-1}(x) = 0.24x$; the formula for the flow rate of the gas**

a. Find $v^{-1}$ of the vent shown. What does this function represent?

b. Determine the speed of air flowing through the vent in meters per second if the flow rate is 425 meters cubed per second. **1770 m/s**

c. Determine the gas flow rate of a circular vent that has a diameter of 1.5 m with a gas stream that is moving at 0.5 meters per second. **$\approx 0.883$ m$^3$**

**68.** **COMMUNICATION** A cellular phone company is having a sale as shown. Assume that the AED 50 rebate is given only after the 10% discount is given.


a. Write a function $r$ for the price of the phone as a function of the original price if only the rebate applies. **$r(x) = x - 50$**

b. Write a function $d$ for the price of the phone as a function of the original price if only the discount applies. **$d(x) = 0.9x$**

c. Find a formula for $T(x) = [r \circ d](x)$ if both the discount and the rebate apply. **$T(x) = 0.9x - 50$**

d. Find $T^{-1}$ and explain what the inverse represents. **See margin.**

e. If the total cost of the phone after the discount and the rebate was AED 49, what was the original price of the phone? **AED 110**

**Use $f(x) = 8x - 4$ and $g(x) = 2x + 6$ to find each of the following.** **69–74. See Chapter 11 Answer Appendix.**

**69.** $[f^{-1} \circ g^{-1}](x)$

**70.** $[g^{-1} \circ f^{-1}](x)$

**71.** $[f \circ g]^{-1}(x)$

**72.** $[g \circ f]^{-1}(x)$

**73.** $(f \cdot g)^{-1}(x)$

**74.** $(f^{-1} \cdot g^{-1})(x)$

**Use $f(x) = x^2 + 1$ with domain $[0, \infty)$ and $g(x) = \sqrt{x - 4}$ to find each of the following.** **75, 77, 79. See margin.**

**75.** $[f^{-1} \circ g^{-1}](x)$

**76.** $[g^{-1} \circ f^{-1}](x)$ **$x + 3$ for $x \geq 1$**

**77.** $[f \circ g]^{-1}(x)$

**78.** $[g \circ f]^{-1}(x)$ **$\sqrt{x^2 + 3}$ for $x \geq 0$**

**79.** $(f \cdot g^{-1})(x)$

**80.** $(f^{-1} \cdot g)(x)$ **$\sqrt{x^2 - 5x + 4}$ for $x \geq 4$**

**81.** **COPIES** Khalaf's Copies charges users AED 0.40 for every minute or part of a minute to use their computer scanner. Suppose you use the scanner for $x$ minutes, where $x$ is any real number greater than 0.

a. Sketch the graph of the function, $C(x)$, that gives the cost of using the scanner for $x$ minutes.

b. What are the domain and range of $C(x)$?

c. Sketch the graph of the inverse of $C(x)$.

d. What are the domain and range of the inverse?

e. What real-world situation is modeled by the inverse?

**a–e. See Chapter 11 Answer Appendix.**

**82.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate inverses of even and odd functions.

a. **GRAPHICAL** Sketch the graphs of three different even functions. Do the graphs pass the horizontal line test?

b. **ANALYTICAL** What pattern can you discern regarding the inverses of even functions? Confirm or deny the pattern algebraically.

c. **GRAPHICAL** Sketch the graphs of three different odd functions. Do the graphs pass the horizontal line test?

d. **ANALYTICAL** What pattern can you discern regarding the inverses of odd functions? Confirm or deny the pattern algebraically.

**a–d. See Chapter 11 Answer Appendix.**

### H.O.T. Problems — Use Higher-Order Thinking Skills

**83.** **REASONING** If $f$ has an inverse and a zero at 6, what can you determine about the graph of $f^{-1}$?

**Sample answer: $f^{-1}(x)$ has $y$-intercept (0, 6).**

**84.** **WRITING IN MATH** Explain what type of restriction on the domain is needed to determine the inverse of a quadratic function and why a restriction is needed. Provide an example. **See Chapter 11 Answer Appendix.**

**85.** **REASONING** *True* or *False*. Explain your reasoning.
*All linear functions have inverse functions.*

**See Chapter 11 Answer Appendix.**

**86.** **CHALLENGE** If $f(x) = x^3 - ax + 8$ and $f^{-1}(23) = 3$, find the value of $a$. **4**

**87.** **REASONING** Can $f(x)$ pass the horizontal line test when $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$? Explain.

**See Chapter 11 Answer Appendix.**

**88.** **REASONING** Why is $\pm$ not used when finding the inverse function of $f(x) = \sqrt{x + 4}$?

**See Chapter 11 Answer Appendix.**

**89.** **WRITING IN MATH** Explain how an inverse of $f$ can exist. Give an example provided that the domain of $f$ is restricted and $f$ does not have an inverse when the domain is unrestricted. **See Chapter 11 Answer Appendix.**





768 | Lesson 11-7 | Inverse Relations and Functions

## Spiral Review

**For each pair of functions, find $f \circ g$ and $g \circ f$. Then state the domain of each composite function.** (Lesson 11-6) **90–92. See margin.**

**90.** $f(x) = x^2 - 9$
$g(x) = x + 4$

**91.** $f(x) = \frac{1}{2}x - 7$
$g(x) = x + 6$

**92.** $f(x) = x - 4$
$g(x) = 3x^2$

**Use the graph of the given parent function to describe the graph of each related function.** (Lesson 11-5) **93–95. See margin.**

**93.** $f(x) = x^2$
**a.** $g(x) = (0.2x)^2$
**b.** $h(x) = (x - 5)^2 - 2$
**c.** $m(x) = 3x^2 + 6$

**94.** $f(x) = x^3$
**a.** $g(x) = |x^3 + 3|$
**b.** $h(x) = -(2x)^3$
**c.** $m(x) = 0.75(x + 1)^3$

**95.** $f(x) = |x|$
**a.** $g(x) = |2x|$
**b.** $h(x) = |x - 5|$
**c.** $m(x) = |3x| - 4$

**96. ADVERTISING** A newspaper surveyed companies on the annual amount of money spent on television commercials and the estimated number of people who remember seeing those commercials each week. A soft-drink manufacturer spends AED 40.1 million a year and estimates 78.6 million people remember the commercials. For a package-delivery service, the budget is AED 22.9 million for 21.9 million people. A telecommunications company reaches 88.9 million people by spending AED 154.9 million. Use a matrix to represent these data. **See margin.**

**Solve each system of equations.**

**97.** $x + 2y + 3z = 5$
$3x + 2y - 2z = -13$
$5x + 3y - z = -11$ **(1, −4, 4)**

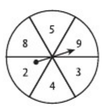
**98.** $7x + 5y + z = 0$
$-x + 3y + 2z = 16$
$x - 6y - z = -18$ **(−2, 2, 4)**

**99.** $x - 3z = 7$
$2x + y - 2z = 11$
$-x - 2y + 9z = 13$ **(10, −7, 1)**

**100. BASEBALL** A batter pops up the ball. Suppose the ball was 3.5 ft above the ground when he hit it straight up with an initial velocity of 80 ft/s. The function $d(t) = 80t - 16t^2 + 3.5$ gives the ball's height above the ground in feet as a function of time $t$ in seconds. How long did the catcher have to get into position to catch the ball after it was hit? **about 5 seconds**

## Skills Review for Standardized Tests

**101. SAT/ACT** What is the probability that the spinner will land on a number that is either even or greater than 5? **D**




A $\frac{1}{6}$
B $\frac{1}{3}$
C $\frac{1}{2}$
D $\frac{2}{3}$
E $\frac{5}{6}$

**102. REVIEW** If $m$ and $n$ are both odd natural numbers, which of the following must be true? **J**

**I.** $m^2 + n^2$ is even.
**II.** $m^2 + n^2$ is divisible by 4.
**III.** $(m + n)^2$ is divisible by 4.

F none
G I only
H I and II only
J I and III only

**103.** Which of the following is the inverse of $f(x) = \frac{3x - 5}{2}$? **A**

A $g(x) = \frac{2x + 5}{3}$
B $g(x) = \frac{3x + 5}{2}$
C $g(x) = 2x + 5$
D $g(x) = \frac{2x - 5}{3}$

**104. REVIEW** A train travels $d$ km in $t$ hours and arrives at its destination 3 hours late. At what average speed, in kilometers per hour, should the train have gone in order to have arrived on time? **H**

F $t - 3$
G $\frac{t - 3}{d}$
H $\frac{d}{t - 3}$
J $\frac{d}{t} - 3$





## 4 Assess

**Name the Math** Ask students to describe how to identify whether a function has an inverse. Use the horizontal line test.

### Additional Answers

**90.** $[f \circ g](x) = x^2 + 8x + 7$ for $\{x \mid x \in \mathbb{R}\}$, $[g \circ f](x) = x^2 - 5$ for $\{x \mid x \in \mathbb{R}\}$

**91.** $[f \circ g](x) = \frac{1}{2}x - 4$ for $\{x \mid x \in \mathbb{R}\}$, $[g \circ f](x) = \frac{1}{2}x - 1$ for $\{x \mid x \in \mathbb{R}\}$

**92.** $[f \circ g](x) = 3x^2 - 4$ for $\{x \mid x \in \mathbb{R}\}$, $[g \circ f](x) = 3x^2 - 24x + 48$ for $\{x \mid x \in \mathbb{R}\}$

**93a.** expanded horizontally

**93b.** translated 5 units to the right and 2 units down

**93c.** expanded vertically, translated 6 units up

**94a.** translated 3 units up, the portion of the graph below the $x$-axis is reflected in the $x$-axis

**94b.** compressed horizontally, reflected in the $x$-axis

**94c.** translated 1 unit to the left, compressed vertically

**95a.** compressed horizontally

**95b.** translated 5 units to the right

**95c.** compressed horizontally translated 4 units down

**96.**

$$\begin{array}{l} \\ \\ \text{Soft-drink} \\ \text{Package delivery} \\ \text{Telecommunications} \end{array} \begin{array}{cc} \text{Budget} & \text{Viewers} \\ \text{(AED million)} & \text{(million)} \\ \left[\begin{array}{c} 40.1 \\ 22.9 \\ 154.9 \end{array}\right. & \left.\begin{array}{c} 78.6 \\ 21.9 \\ 88.9 \end{array}\right] \end{array}$$

## Differentiated Instruction BL

**Extension** Does the function $f(x) = [\![x]\!]$ have an inverse that is a function? Explain. No; the graph does not pass the horizontal line test. Therefore, the inverse has elements in the domain paired with more than one element in the range.

# 1 Focus

**Objective** Use a graphing calculator and parametric equations to graph inverses on the calculator.

### Teaching Tip

To help students think in terms of parametric equations, review the concept of parametric equations and then complete the following example. Given $y = t^3$ and $x = t$, create a table with three columns: one for $t$, a second for $x$, and a third for $y$. Have students write several points for $t = 1, 2, 3$, and so on.

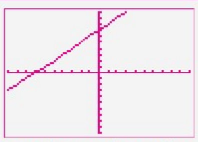
# 2 Teach

### Working in Cooperative Groups

Form pairs of students, matching those who are proficient with the graphing calculator with students who are not proficient. As students work through examples, have them alternate between the roles of graphing calculator operator and graphing calculator coach.

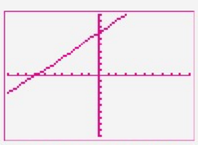
### Additional Answers

**2a.**



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1
$t$: $[-10, 10]$; $t$step: 1

**2b.**



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1
$t$: $[-10, 10]$; $t$step: 1

EXTEND 11-7

# Graphing Technology Lab
# Graphing Inverses using Parametric Equations

## Objective

- Use a graphing calculator and parametric equations to graph inverses on the calculator.

**New Vocabulary**
parametric equations

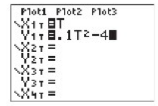
**Parametric equations** are equations that can express the position of an object as a function of time. The basic premise of parametric equations is the introduction of an extra variable $t$, called a *parameter*. For example, $y = x + 4$ can be expressed parametrically using $x = t$ and $y = t + 4$.

**Activity 1** Parametric Graph

Graph $x = t$, $y = 0.1t^2 - 4$.

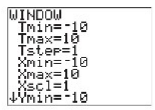
**Step 1** Set the mode. In the MODE menu, select par and simul. This allows the equations to be graphed simultaneously.


**Step 2** Enter the parametric equations. In parametric form, X,T,θ,n will use $t$ instead of $x$.

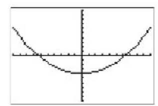



**StudyTip**
Standard Window You can use ZoomStandard to set the window in standard form.

**Step 3** Set the window as shown.




**Step 4** Graph the equations. Notice that the graph looks like $y = 0.1x^2 - 4$ but is traced from $t = -10$ to $t = 10$.



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1
$t$: $[-10, 10]$; tstep: 1

## Exercises

1. **REASONING** Graph the equations using Tstep = 10, 5, 0.5, and 0.1. How does this affect the way the graph is shown? **Sample answer: As the Tstep level decreases, the graph becomes more of a smooth curve.**

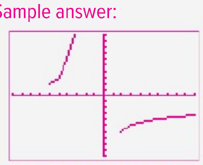
2. In this problem, you will investigate the relationship between $x$, $y$, and $t$.
   a. Graph $X_{1T} = t - 3$, $Y_{1T} = t + 4$ in the standard viewing window. **a–b. See margin.**
   b. Replace the equations in part **a** with $X_{1T} = t$, $Y_{1T} = t + 7$ and graph.
   c. What do you notice about the two graphs? **They are identical.**
   d. **REASONING** What conclusions can you make about the relationship between $x$, $y$, and $t$? In other words, how do you think the second set of parametric equations was formed using the first set?

**2d. Sample answer: The second set of parametric equations was formed by substituting for $t$.**





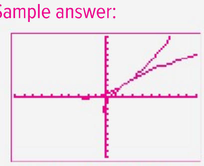
**5.** Sample answer:



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1
$t$: $[0, 10]$; $t$step: 1

$D = \{t \mid t \geq 0\}$

**6.** Sample answer:



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1
$t$: $[-0.5, 10]$; $t$step: 1

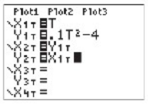
$D = \{t \mid t \geq -0.5\}$

One benefit of parametric equations is the ability to graph inverses without determining them.

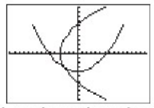
**Activity 2** Graph an Inverse

**Graph the inverse of $x = t$, $y = 0.1t^2 - 4$.**

**Step 1** Enter the given equations as $X_{1T}$ and $Y_{1T}$. To graph the inverse, set $X_{2T} = Y_{1T}$ and $Y_{2T} = X_{1T}$. These are found in the VARS menu. Select Y-Vars, parametric, $X_{1T}$.




**Step 2** Graph the relation and the inverse. You can use ZSquare to see the symmetry of the two graphs more clearly.



[−15, 15] scl: 1 by [−10, 10] scl: 1
$t$: [−10, 10]; tstep: 1

### Exercises

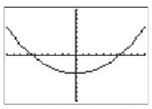
3. **REASONING** What needs to be true about the ordered pairs of each graph in Activity 2? **Sample answer: $(x, y)$ for one graph is identical to $(y, x)$ for the other.**

4. **REASONING** Does the graph of $x = t$, $y = 0.1t^2 - 4$ represent a one-to-one function? Explain. **No; sample answer: The function fails the horizontal line test.**

**Activity 3** Domains and One-to-One Functions

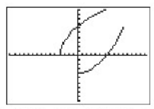
**Limit the domain of $x = t$, $y = 0.1t^2 - 4$ in order to make it one-to-one.**

**Step 1** The minimum of the graph is located at $t = 0$. We can produce a one-to-one function for $t$ values $0 \le t \le 10$.



[−10, 10] scl: 1 by [−10, 10] scl: 1
$t$: [−10, 10]; tstep: 1

**Step 2** Change Tmin from −10 to 0 in order to limit the domain. Graph the one-to-one function and its inverse.



[−15, 15] scl: 1 by [−10, 10] scl: 1
$t$: [0, 10]; tstep: 1

**StudyTip**

**Symmetry** You may need to use the TRACE feature and adjust Tstep in order to locate the axis of symmetry.

### Exercises

**Graph each function. Then graph the inverse function and indicate the limited domain if necessary. 5–10. See margin.**

5. $x = t - 6$, $y = t^2 + 2$
6. $x = 3t - 1$, $y = t^2 + t$
7. $x = 3 - 2t$, $y = t^2 - 2t + 1$
8. $x = 2t^2 + 3$, $y = \sqrt{t}$
9. $x = 4t$, $y = \sqrt{t + 2}$
10. $x = t - 8$, $y = t^3$

11. **CHALLENGE** Consider a quintic function with two relative maxima and two relative minima. Into how many different one-to-one functions can this function be separated if each separation uses the largest interval possible? **5**





**Practice** Have students complete Exercises 1–5, 8, and 11.

## 3 Assess

### Formative Assessment

Use Exercises 6, 7, 9, and 10 to assess students' understanding of parametric equations.

### From Concrete to Abstract

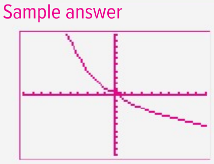
Given the following sets of parametric equations, have students identify the point in time at which one $y$-value becomes greater than the other.

$X_1 = t$; $Y_1 = t^2$
$X_2 = t$; $Y_2 = 2t + 2$

Set $Y_1$ equal to $Y_2$ and solve for $t$. The solution is $t = \sqrt{3} + 1$. Therefore, for $t$ on the interval $(0, \sqrt{3} + 1)$, $Y_1 < Y_2$. For $t$ on the interval $(\sqrt{3} + 1, \infty)$, $Y_1 > Y_2$. The change in the relation between $y$ values occurs at $t = \sqrt{3} + 1$.

### Additional Answers

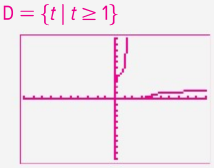
**7.** Sample answer



[−10, 10] scl: 1 by [−10, 10] scl: 1
$t$: [1, 10]; $t$step: 1

$D = \{t \mid t \ge 1\}$

**8.**



[−10, 10] scl: 1 by [−10, 10] scl: 1
$t$: [0, 10]; $t$step: 1

D: $\{t \mid t \ge 0\}$

**9.**

[−10, 10] scl: 1 by [−10, 10] scl: 1
$t$: [−2, 10]; $t$step: 1

D: $\{t \mid t \ge -2\}$

**10.**

[−10, 10] scl: 1 by [−10, 10] scl: 1
$t$: [−10, 10]; $t$step: 1

## Formative Assessment

**KeyVocabulary** The page references after each word denote where that term was first introduced. If students have difficulty answering Questions 1–10, remind them that they can use these page references to refresh their memories about the vocabulary.

# CHAPTER 11 Study Guide and Review

## Study Guide

### KeyConcepts

**Functions** (Lesson 11-1)

- Common subsets of the real numbers are integers, rational numbers, irrational numbers, whole numbers, and natural numbers.
- A function *f* is a relation that assigns each element in the domain exactly one element in the range.
- The graph of a function passes the vertical line test.

**Analyzing Graphs of Functions and Relations** (Lesson 11-2)

- Graphs may be symmetric with respect to the *y*-axis, the *x*-axis, and the origin.
- An even function is symmetric with respect to the *y*-axis. An odd function is symmetric with respect to the origin.

**Continuity, End Behavior, and Limits** (Lesson 11-3)

- If the value of $f(x)$ approaches a unique value $L$ as $x$ approaches $c$ from either side, then the limit of $f(x)$ as $x$ approaches $c$ is $L$. It is written $\lim_{x \to c} f(x) = L$.
- A function may be discontinuous because of infinite discontinuity, jump discontinuity, or removable discontinuity.

**Extrema and Average Rate of Change** (Lesson 11-4)

- A function can be described as increasing, decreasing, or constant.
- Extrema of a function include relative maxima and minima and absolute maxima and minima.
- The average rate of change between two points can be represented by $m_{sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

**Parent Functions and Transformations** (Lesson 11-5)

Transformations of parent functions include translations, reflections, and dilations.

**Operations and Composition of Functions** (Lesson 11-6)

The sum, difference, product, quotient, and composition of two functions form new functions.

**Inverse Relations and Functions** (Lesson 11-7)

- Two relations are inverse relations if and only if one relation contains the element $(b, a)$ whenever the other relation contains the element $(a, b)$.
- Two functions, $f$ and $f^{-1}$, are inverse functions if and only if $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$.

### KeyVocabulary

| | |
|---|---|
| composition | line symmetry |
| constant | maximum |
| continuous function | minimum |
| decreasing function | nonremovable discontinuity |
| dilation | odd function |
| discontinuous function | one-to-one |
| end behavior | parent function |
| even function | piecewise-defined function |
| extrema | point symmetry |
| function | reflection |
| increasing | roots |
| interval notation | translation |
| inverse function | zero function |
| inverse relation | zeros |
| limit | |

### VocabularyCheck

**State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.**

1. A <u>function</u> assigns every element of its domain to exactly one element of its range. **true**
2. Graphs that have <u>point symmetry</u> can be rotated 180° with respect to a point and appear unchanged. **true**
3. An <u>odd function</u> has a point of symmetry. **true**
4. The graph of a <u>continuous function</u> has no breaks or holes. **true**
5. The <u>limit</u> of a graph describes approaching a value without necessarily ever reaching it. **false; end behavior**
6. A function $f(x)$ with values that decrease as $x$ increases is a <u>decreasing</u> function. **true**
7. The <u>extrema</u> of a function can include relative maxima or minima. **true**
8. The <u>translation</u> of a graph produces a mirror image of the graph with respect to a line. **false; reflection**
9. A <u>one-to-one function</u> passes the horizontal line test. **true**
10. One-to-one functions have <u>line symmetry</u>. **false; inverse functions**

# Lesson-by-Lesson Review

## 11-1 Functions

Determine whether each relation represents $y$ as a function of $x$.

**11.** $3x - 2y = 18$ **function**

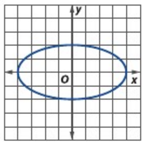
**12.** $y^3 - x = 4$ **function**

**13.**

| $x$ | $y$ |
|---|---|
| 5 | 7 |
| 7 | 9 |
| 9 | 11 |
| 11 | 13 |

**function**

**14.**




**not a function**

Let $f(x) = x^2 - 3x + 4$. Find each function value.

**15.** $f(5)$ **14**

**16.** $f(-3x))$ $\mathbf{9x^2 + 9x + 4}$

State the domain of each function.

**17.** $f(x) = 5x^2 - 17x + 1$ **D = {x | x ∈ ℝ}**

**18.** $g(x) = \sqrt{6x - 3}$ **D = [x | x ≥ 0.5, x ∈ ℝ]**

**19.** $h(a) = \frac{5}{a + 5}$ **D = {a | a ≠ −5, a ∈ ℝ}**

**20.** $v(x) = \frac{x}{x^2 - 4}$ **D = {x | x ≠ ±2, x ∈ ℝ}**

### Example 1

**Determine whether $y^2 - 8 = x$ represents $y$ as a function of $x$.**

Solve for $y$.

$y^2 - 8 = x$ Original equation

$y^2 = x + 8$ Add 8 to each side.

$y = \pm\sqrt{x + 8}$ Take the square root of each side.

This equation does not represent $y$ as a function of $x$ because for any $x$-value greater than $-8$, there will be two corresponding $y$-values.

### Example 2

**Let $g(x) = -3x^2 + x - 6$. Find $g(2)$.**

Substitute 2 for $x$ in the expression $-3x^2 + x - 6$.

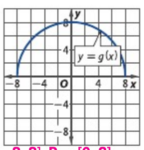
$g(2) = -3(2)^2 + 2 - 6$ $x = 2$

$= -12 + 2 - 6$ or $-16$ Simplify.

## 11-2 Analyzing Graphs of Functions and Relations

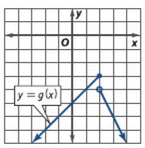
Use the graph of $g$ to find the domain and range of each function.

**21.**




**D = [−8, 8], R = [0, 8]**

**22.**




**D = {x | x ∈ ℝ}, R = (−∞, −3)**

Find the $y$-intercept(s) and zeros for each function.

**23.** $f(x) = 4x - 9$ $\mathbf{-9; \frac{9}{4}}$

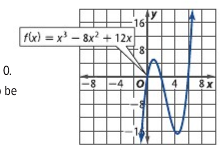
**24.** $f(x) = x^2 - 6x - 27$ **−27; −3, 9**

**25.** $f(x) = x^3 - 16x$ **0; 0, 4, −4**

**26.** $f(x) = \sqrt{x + 2} - 1$ $\mathbf{\sqrt{2} - 1; -1}$

### Example 3

**Use the graph of $f(x) = x^3 - 8x^2 + 12x$ to find its $y$-intercept and zeros. Then find these values algebraically.**




**Estimate Graphically**

It appears that $f(x)$ intersects the $y$-axis at $(0, 0)$, so the $y$-intercept is 0.

The $x$-intercepts appear to be at about 0, 2, and 6.

**Solve Algebraically**

Find $f(0)$.

$f(0) = (0)^3 - 8(0)^2 + 12(0)$ or 0

The $y$-intercept is 0.

Factor the related equation.

$x(x^2 - 8x + 12) = 0$

$x(x - 6)(x - 2) = 0$

The zeros of $f$ are 0, 6, and 2.





## Lesson-by-Lesson Review

**Intervention** If the given examples are not sufficient to review the topics covered by the questions, remind students that the page references tell them where to review that topic in their textbooks.

## Additional Answers

**27.** continuous at $x = 4$; The function is defined when $x = 4$. The function approaches 4 when $x$ approaches 4 from both sides, and $f(4) = 4$.

**28.** continuous at $x = 10$; The function is defined when $x = 10$. The function approaches 4 when $x$ approaches 10 from both sides

**29.** continuous at $x = 0$; The function is defined when $x = 0$. The function approaches 0 when $x$ approaches 0 from both sides, and $f(0) = 0$. continuous at $x = 7$; The function is defined when $x = 7$. The function approaches 0.5 when $x$ approaches 7 from both sides, and $f(7) = 0.5$.

**30.** discontinuous at $x = 2$; The function is not defined when $x = 2$. It is an infinite discontinuity. The function is continuous at $x = 4$; The function is defined when $x = 4$. The function approaches $\frac{1}{3}$ when $x$ approaches 4 from both sides, and $f(4) = \frac{1}{3}$.

**31.** continuous at $x = 1$; The function is defined when $x = 1$. The function approaches 2 when $x$ approaches 1 from both sides, and $f(1) = 2$.

**32.** From the graph, it appears that as $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to \infty$.

**33.** From the graph, it appears that as $x \to \infty$, $f(x) \to 0$; as $x \to -\infty$, $f(x) \to 0$.

**34.** $f$ is increasing on $(-\infty, -0.5)$, decreasing on $(-0.5, 0.5)$, and increasing on $(0.5, \infty)$; relative maximum at $(-0.5, 3.5)$ and relative minimum at $(0.5, 2.5)$.

**35.** $f$ is decreasing on $(-\infty, -3)$, increasing on $(-3, -1.5)$, decreasing on $(-1.5, 0.5)$, and increasing on $(0.5, \infty)$; relative minimum at $(-3, 3)$, relative maximum at $(-1.5, 6)$ and relative minimum at $(0.5, -7)$.

# CHAPTER 11 Study Guide and Review *Continued*

## 11-3 Continuity, End Behavior, and Limits

Determine whether each function is continuous at the given $x$-value(s). Justify using the continuity test. If discontinuous, identify the type of discontinuity as infinite, jump, or removable. **27–31. See margin.**

**27.** $f(x) = x^2 - 3x$; $x = 4$.

**28.** $f(x) = \sqrt{2x - 4}$; $x = 10$

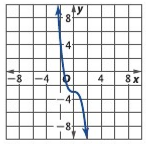
**29.** $f(x) = \frac{x}{x+7}$; $x = 0$ and $x = 7$

**30.** $f(x) = \frac{x}{x^2 - 4}$; $x = 2$ and $x = 4$

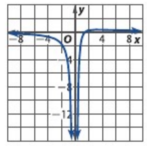
**31.** $f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$; $x = 1$

Use the graph of each function to describe its end behavior. **32–33. See margin.**

**32.**



**33.**



### Example 4

Determine whether $f(x) = \frac{1}{x-4}$ is continuous at $x = 0$ and $x = 4$. Justify your answer using the continuity test. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.

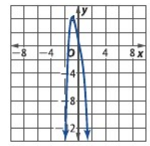
$f(0) = -0.25$, so $f$ is defined at 0. The function values suggest that as $f$ gets closer to $-0.25$ $x$ gets closer to 0.

| $x$ | −0.1 | −0.01 | 0 | 0.01 | 0.1 |
|---|---|---|---|---|---|
| $f(x)$ | −0.244 | −0.249 | −0.25 | −0.251 | −0.256 |

Because $\lim_{x \to 0} f(x)$ is estimated to be $-0.25$ and $f(0) = -0.25$, we can conclude that $f(x)$ is continuous at $x = 0$.
Because $f$ is not defined at 4, $f$ is not continuous at 4.

### Example 5

Use the graph of $f(x) = -2x^4 - 5x + 1$ to describe its end behavior.

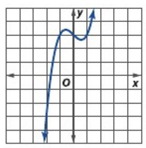


Examine the graph of $f(x)$.
As $x \to \infty$, $f(x) \to -\infty$.
As $x \to -\infty$, $f(x) \to -\infty$.

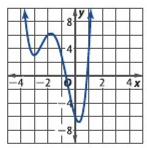
## 11-4 Extreme and Average Rate of Change

Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Then estimate to the nearest 0.5 unit, and classify the extrema for the graph of each function. **34–35. See margin.**

**34.**



**35.**



Find the average rate of change of each function on the given interval.

**36.** $f(x) = -x^3 + 3x + 1$; $[0, 2]$ **−1**

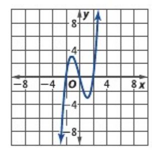
**37.** $f(x) = x^2 + 2x + 5$; $[-5, 3]$ **0**

### Example 6

Use the graph of $f(x) = x^3 - 4x$ to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Then estimate to the nearest 0.5 unit and classify the extrema for the graph of each function.

From the graph, we can estimate that $f$ is increasing on $(-\infty, -1)$, decreasing on $(-1, 1)$, and increasing on $(1, \infty)$.

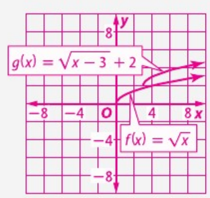
We can estimate that $f$ has a relative maximum at $(-1, 3)$ and a relative minimum at $(1, -3)$.



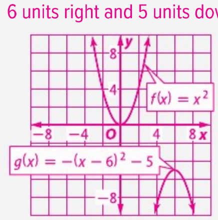




**38.** $f(x) = \sqrt{x}$; $g(x)$ is the graph of $f(x)$ translated 3 units right and 2 units up.




**39.** $f(x) = x^2$; $g(x)$ is the graph of $f(x)$ reflected in the $x$-axis and translated 6 units right and 5 units down.



## 11-5 Parent Functions and Transformations

Identify the parent function $f(x)$ of $g(x)$, and describe how the graphs of $g(x)$ and $f(x)$ are related. Then graph $f(x)$ and $g(x)$ on the same axes. **38–41. See margin.**

**38.** $g(x) = \sqrt{x-3} + 2$

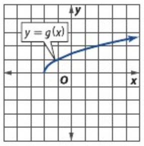
**39.** $g(x) = -(x-6)^2 - 5$

**40.** $g(x) = \frac{1}{2(x+7)}$

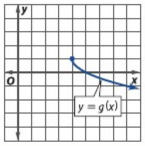
**41.** $g(x) = \frac{1}{4}[\![x]\!] + 3$

Describe how the graphs of $f(x) = \sqrt{x}$ and $g(x)$ are related. Then write an equation for $g(x)$.

**42.**




**43.**



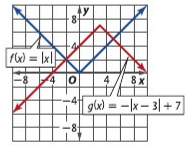

**42. The graph is translated 2 units left; $g(x) = \sqrt{x+2}$.**

**43. The graph is reflected in the $x$-axis and translated 4 units right and 1 unit up; $g(x) = -\sqrt{x-4} + 1$.**

### Example 7

Identify the parent function $f(x)$ of $g(x) = -|x-3| + 7$, and describe how the graphs of $g(x)$ and $f(x)$ are related. Then graph $f(x)$ and $g(x)$ on the same axes.

The parent function for $g(x)$ is $f(x) = |x|$. The graph of $g$ will be the same as the graph of $f$ reflected in the $x$-axis, translated 3 units to the right, and translated 7 units up.




## 11-6 Function Operations and Composition of Functions

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. State the domain of each new function. **44–47. See margin.**

**44.** $f(x) = x + 3$
$g(x) = 2x^2 + 4x - 6$

**45.** $f(x) = 4x^2 - 1$
$g(x) = 5x - 1$

**46.** $f(x) = x^3 - 2x^2 + 5$
$g(x) = 4x^2 - 3$

**47.** $f(x) = \frac{1}{x}$
$g(x) = \frac{1}{x^2}$

For each pair of functions, find $[f \circ g](x)$, $[g \circ f](x)$, and $[f \circ g](2)$.

**48.** $f(x) = 4x - 11$; $g(x) = 2x^2 - 8$ **$8x^2 - 43$; $32x^2 - 176x + 234$; $-11$**

**49.** $f(x) = x^2 + 2x + 8$; $g(x) = x - 5$ **$x^2 - 8x + 23$; $x^2 + 2x + 3$; $11$**

**50.** $f(x) = x^2 - 3x + 4$; $g(x) = x^2$ **$x^4 - 3x^2 + 4$; $x^4 - 6x^3 + 17x^2 - 24x + 16$; $8$**

Find $f \circ g$. **51–52. See margin.**

**51.** $f(x) = \frac{1}{x-3}$
$g(x) = 2x - 6$

**52.** $f(x) = \sqrt{x-2}$
$g(x) = 6x - 7$

### Example 8

Given $f(x) = x^3 - 1$ and $g(x) = x + 7$, find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$. State the domain of each new function.

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (x^3 - 1) + (x + 7)\\ &= x^3 + x + 6\end{aligned}$$

The domain of $(f+g)(x)$ is $(-\infty, \infty)$.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= (x^3 - 1) - (x + 7)\\ &= x^3 - x - 8\end{aligned}$$

The domain of $(f-g)(x)$ is $(-\infty, \infty)$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= (x^3 - 1)(x + 7)\\ &= x^4 + 7x^3 - x - 7\end{aligned}$$

The domain of $(f \cdot g)(x)$ is $(-\infty, \infty)$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ or } \frac{x^3 - 1}{x + 7}$$

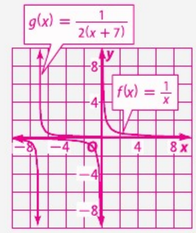
The domain of $\left(\frac{f}{g}\right)(x)$ is $D = (-\infty, -7) \cup (-7, \infty)$.



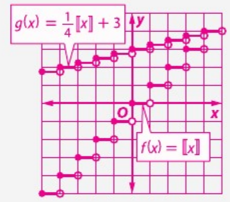


## Additional Answers

**40.** $f(x) = \frac{1}{x}$; $g(x)$ is the graph of $f(x)$ translated 7 units left, and is compressed vertically by a factor of $\frac{1}{2}$.




**41.** $f(x) = [\![x]\!]$; $g(x)$ is the graph of $f(x)$ compressed vertically by a factor of $\frac{1}{4}$ and translated 3 units up.




**44.** $(f+g)(x) = 2x^2 + 5x - 3$; $D = (-\infty, \infty)$; $(f-g)(x) = -2x^2 - 3x + 9$; $D = (-\infty, \infty)$; $(f \cdot g)(x) = 2x^3 + 10x^2 + 6x - 18$; $D = (-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{1}{2(x-1)}$; $D = (-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

**45.** $(f+g)(x) = 4x^2 + 5x - 2$; $D = (-\infty, \infty)$; $(f-g)(x) = 4x^2 - 5x$; $D = (-\infty, \infty)$; $(f \cdot g)(x) = 20x^3 - 4x^2 - 5x + 1$; $D = (-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{4x^2 - 1}{5x - 1}$; $D = \left(-\infty, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \infty\right)$

**46.** $(f+g)(x) = x^3 + 2x^2 + 2$; $D = (-\infty, \infty)$; $(f-g)(x) = x^3 - 6x^2 + 8$; $D = (-\infty, \infty)$; $(f \cdot g)(x) = 4x^5 - 8x^4 - 3x^3 + 26x^2 - 15$; $D = (-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x^3 - 2x^2 + 5}{4x^2 - 3}$; $D = \left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$

**47.** $(f+g)(x) = \frac{x+1}{x^2}$; $D = (-\infty, 0) \cup (0, \infty)$;

$(f-g)(x) = \frac{x-1}{x^2}$; $D = (-\infty, 0) \cup (0, \infty)$;

$(f \cdot g)(x) = \frac{1}{x^3}$; $D = (-\infty, 0) \cup (0, \infty)$;

$\left(\frac{f}{g}\right)(x) = x$; $D = (-\infty, 0) \cup (0, \infty)$

**51.** $[f \circ g](x) = \frac{1}{2x - 9}$ for $x \neq \frac{9}{2}$

**52.** $[f \circ g](x) = \sqrt{6x - 9}$ for $x \geq \frac{3}{2}$

## Additional Answers

**57.** $f^{-1}(x) = \sqrt[3]{x+2}$

**58.** $g^{-1}(x) = -\frac{1}{4}x + 2$

**59.** $h^{-1}(x) = \frac{1}{4}x^2 - 3, x \geq 0$

**60.** $f^{-1}(x) = \frac{-2x}{x-1}, x \neq 1$

**64a.** Sample answer: The number of home runs decreased, then increased and because 23 is not the smallest number of home runs.

**64c.** Sample answer: There were fewer home runs in 2012 than in 2007.

**67a.** $A(x) = 6.4516x$ cm$^2$

**67b.** $A^{-1}(x) = \frac{1}{6.4516}x$ in$^2$



## 11-7 Inverse Relations and Functions

Graph each function using a graphing calculator, and apply the horizontal line test to determine whether its inverse function exists. Write *yes* or *no*.

**53.** $f(x) = |x| + 6$ **no**

**54.** $f(x) = x^3$ **yes**

**55.** $f(x) = -\frac{3}{x+6}$ **yes**

**56.** $f(x) = x^3 - 4x^2$ **no**

Find the inverse function and state any restrictions on the domain. **57–60. See margin.**

**57.** $f(x) = x^3 - 2$

**58.** $g(x) = -4x + 8$

**59.** $h(x) = 2\sqrt{x+3}$

**60.** $f(x) = \frac{x}{x+2}$

### Example 9

Find the inverse function of $f(x) = \sqrt{x} - 3$ and state any restrictions on its domain.

Note that $f$ has domain $[0, \infty)$ and range $[-3, \infty)$. Now find the inverse relation of $f$.

$y = \sqrt{x} - 3$ Replace $f(x)$ with $y$.

$x = \sqrt{y} - 3$ Interchange $x$ and $y$.

$x + 3 = \sqrt{y}$ Add 3 to each side.

$(x+3)^2 = y$ Square each side. Note that $D = (-\infty, \infty)$ and $R = [0, \infty)$.

The domain of $y = (x+3)^2$ does not equal the range of $f$ unless restricted to $[-3, \infty)$. So, $f^{-1}(x) = (x+3)^2$ for $x \geq -3$.

## Applications and Problem Solving

**61a, c. See Chapter 11 Answer Appendix.**

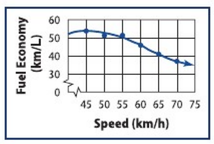
**61.** **CELL PHONES** Basic Mobile offers a cell phone plan that charges AED 39.99 per month. Included in the plan are 500 daytime minutes that can be used Sunday through Thursday between 7 A.M. and 7 P.M. Users are charged AED 0.20 per minute for every daytime minute over 500 used. (Lesson 11-1)

a. Write a function $p(x)$ for the cost of a month of service during which you use $x$ daytime minutes.

b. How much will you be charged if you use 450 daytime minutes? 550 daytime minutes?

c. Graph $p(x)$. **AED 39.99; AED 49.99**

**62.** **AUTOMOBILES** The fuel economy for a hybrid car at various highway speeds is shown. (Lesson 11-2)




**Sample answer: about 51 km/L**

a. Approximately what is the fuel economy for the car when traveling 50 km/h?

b. At approximately what speed will the car's fuel economy be less than 40 km/L?

**Sample answer: about 67 km/h or faster**

**63.** **SALARIES** After working for a company for five years, Ms. Muna was given a promotion. She is now earning AED 1500 per month more than her previous salary. Will a function modeling her monthly income be a continuous function? Explain. (Lesson 11-3)

**No; sample answer: At the time of her promotion, her income had a jump discontinuity.**

**64.** **BASEBALL** The table shows the number of home runs by a baseball player in each of the first 5 years he played professionally. (Lesson 11-4)

| Year | 2004 | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|---|
| Number of Home Runs | 5 | 36 | 23 | 42 | 42 |

a. Explain why 2006 represents a relative minimum.

b. Suppose the average rate of change of home runs between 2008 and 2011 is 5 home runs per year. How many home runs were there in 2011? **57 home runs**

c. Suppose the average rate of change of home runs between 2007 and 2012 is negative. Compare the number of home runs in 2007 and 2012. **64a, c. See margin.**

**65.** **PHYSICS** A stone is thrown horizontally from the top of a cliff. The velocity of the stone measured in meters per second after $t$ seconds can be modeled by $v(t) = -\sqrt{(9.8t)^2 + 49}$. The speed of the stone is the absolute value of its velocity. Draw a graph of the stone's speed during the first 6 seconds. (Lesson 11-5)

**See Chapter 11 Answer Appendix.**

**66.** **FINANCIAL LITERACY** A department store advertises AED 40 off the price of any pair of jeans. How much will a pair of jeans cost if the original price is AED 220 and there is 8.5% sales tax? (Lesson 11-6) **AED 195.30**

**67a–b. See margin.**

**67.** **MEASUREMENT** One inch is approximately equal to 2.54 cm. (Lesson 11-7)

a. Write a function $A(x)$ that will convert the area $x$ of a rectangle from in$^2$ to cm$^2$.

b. Write a function $A^{-1}(x)$ that will convert the area $x$ of a rectangle from cm$^2$ to in$^2$.



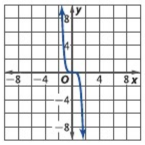


776 | Chapter 11 | Study Guide and Review

# Practice Test



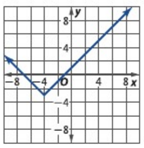
Determine whether the given relation represents $y$ as a function of $x$.

1. $x = y^2 - 5$ **not a function**
2. 

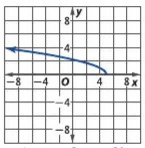
**function**
3. $y = \sqrt{x^2 + 3}$ **function**

4. **PARKING** The cost of parking a car downtown is AED 0.75 per 30 minutes for a maximum of AED 4.50. Parking is charged per second.
   a. Write a function for $c(x)$, the cost of parking a car for $x$ hours. $c(x) = \begin{cases} 1.5x & \text{if } 0 \le x \le 3 \\ 4.5 & \text{if } x > 3 \end{cases}$
   b. Find $c(2.5)$. **AED 3.75**
   c. What is the domain for $c(x)$? Explain your reasoning. **$D = [0, \infty)$; Sample answer: The number of hours must be greater than or equal to 0.**

State the domain and range of each function.

5. 

**$D = (-\infty, \infty)$, $R = [-3, \infty)$**

6. 

**$D = (-\infty, 5]$, $R = [0, \infty)$**

Find the $y$-intercept(s) and zeros for each function.

7. $f(x) = 4x^2 - 8x - 12$ **−12; −1, 3**
8. $f(x) = x^3 + 4x^2 + 3x$ **0; −3, −1, 0**

9. **MULTIPLE CHOICE** Which relation is symmetric with respect to the $x$-axis? **D**
   A $-x^2 - yx = 2$
   B $x^3y = 8$
   C $y = |x|$
   D $-y^2 = -4x$

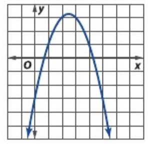
Determine whether each function is continuous at $x = 3$. If discontinuous, identify the type of discontinuity as *infinite*, *jump*, or *removable*.

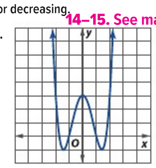
10. $f(x) = \begin{cases} 2x & \text{if } x < 3 \\ 9 - x & \text{if } x \ge 3 \end{cases}$ **continuous**
11. $f(x) = \dfrac{x - 3}{x^2 - 9}$ **discontinuous; removable discontinuity**

Find the average rate of change for each function on the interval $[-2, 6]$.

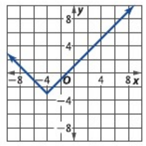
12. $f(x) = -x^4 + 3x$ **−157**
13. $f(x) = \sqrt{x + 3}$ $\mathbf{\frac{1}{4}}$

Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing or decreasing. **14–15. See margin.**

14. 


15. 


16. **MULTIPLE CHOICE** Which function is shown in the graph? **H**




   F $f(x) = |x - 4| - 3$
   G $f(x) = |x - 4| + 3$
   H $f(x) = |x + 4| - 3$
   J $f(x) = |x + 4| + 3$

Identify the parent function $f(x)$ of $g(x)$. Then sketch the graph of $g(x)$.

17. $g(x) = -(x + 3)^3$
18. $g(x) = |x^2 - 4|$ **17–18. See margin.**

Given $f(x) = x - 6$ and $g(x) = x^2 - 36$, find each function and its domain. **19–20. See margin.**

19. $\left(\frac{f}{g}\right)(x)$
20. $[g \circ f](x)$

21. **TEMPERATURE** In most countries, temperature is measured in degrees Celsius. The equation that relates degrees Fahrenheit with degrees Celsius is $F = \frac{9}{5}C + 32$.
   a. Write $C$ as a function of $F$. $\mathbf{C = \frac{5}{9}(F - 32)}$
   b. Find two functions $f$ and $g$ such that $C = [f \circ g](F)$. **Sample answer: $f(x) = \frac{5}{9}x$; $g(x) = x - 32$**

Determine whether $f$ has an inverse function. If it does, find the inverse function and state any restrictions on its domain.

22. $f(x) = (x - 2)^3$ **yes; $f^{-1}(x) = \sqrt[3]{x} + 2$**
23. $f(x) = \dfrac{x + 3}{x - 8}$ **yes; $f^{-1}(x) = \dfrac{8x + 3}{x - 1}$; $x \ne 1$**
24. $f(x) = \sqrt{4 - x}$ **yes; $f^{-1}(x) = 4 - x^2$; $x \ge 0$**
25. $f(x) = x^2 - 16$ **no**



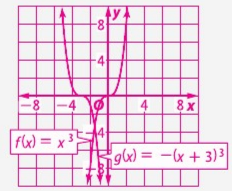


## Additional Answers

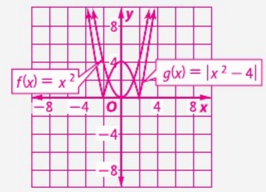
14. $f$ is increasing on $(-\infty, 2.5)$ and decreasing on $(2.5, \infty)$.

15. $f$ is decreasing on $(-\infty, -1.5)$, increasing on $(-1.5, 0)$, decreasing on $(0, 1.5)$, and increasing on $(1.5, \infty)$.

17. $f(x) = x^3$




18. $f(x) = x^2$




19. $\left(\frac{f}{g}\right)(x) = \dfrac{1}{x + 6}$ for $x \ne -6$ or $x \ne 6$

20. $[f \circ g](x) = x^2 - 12x$ for $x \in \mathbb{R}$

## 1 Focus

**Objective** Approximate the rate of change of a function at a point.

### Teaching Tip

Draw a parabola on the board and have students identify the tangent line at the vertex. After drawing the line, pick pairs of points on either side of the vertex and use a ruler to draw secant lines. Have students describe the change to the secant as the points converge on the vertex. The secant eventually looks like the tangent line.

## 2 Teach

### Working in Cooperative Groups

Have students work in groups of three or four, mixing abilities. Have groups complete Activities 1–3 and Analyze the Results Exercises 1–9.

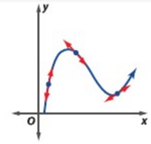
CHAPTER 11

# Connect to AP Calculus
# Rate of Change at a Point

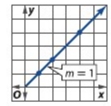
**Objective**

- Approximate the rate of change of a function at a point.

*Differential calculus* is a branch of calculus that focuses on the rates of change of functions at individual points. You have learned to calculate the constant rate of change, or slope, for linear functions and the average rate of change for nonlinear functions. Using differential calculus, you can determine the *exact* rate of change of any function at a single point, as represented by the slopes of the tangent lines in the figure at the right.




The constant rate of change for a linear function not only represents the slope of the graph between two points, but also the *exact* rate of change of the function at each of its points. For example, notice in the figure at the right that the slope $m$ of the function is 1. This value also refers to the *exact* rate at which this function is changing at any point in its domain. This is the focus of differential calculus.




The average rate of change for nonlinear functions is represented by the slope of a line created by any two points on the graph of the function. This line is called a *secant line*. This slope is not the *exact* rate of change of the function at any one point. We can, however, use this process to give us an approximation for that instantaneous rate of change.

**Activity 1** Approximate Rate of Change

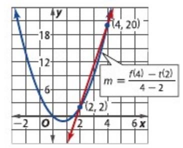
**Approximate the rate of change of $f(x) = 2x^2 - 3x$ at $x = 2$.**

**Step 1** Graph $f(x) = 2x^2 - 3x$, and plot the point $P = (2, f(2))$.

**Step 2** Draw a secant line through $P = (2, f(2))$ and $Q = (4, f(4))$.

**Step 3** Calculate the average rate of change $m$ for $f(x)$ using $P$ and $Q$, as shown in the figure.

**Step 4** Repeat Steps 1–3 four more times. Use $Q = (3, f(3))$, $Q = (2.5, f(2.5))$, $Q = (2.25, f(2.25))$, and $Q = (2.1, f(2.1))$.




**Analyze the Results**

1. As $Q$ approaches $P$, what does the average rate of change $m$ appear to approach? 5

2. Using a secant line to approximate rate of change at a point can produce varying results. Make a conjecture as to when this process will produce accurate approximations.

**2. Sample answer: Secant lines will produce accurate approximations when the two points on the graph are relatively close to one another.**

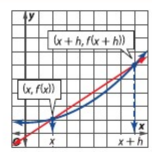
In differential calculus, we express the formula for average rate of change in terms of $x$ and the horizontal distance $h$ between the two points that determine the secant line.

**Activity 2** Approximate Rate of Change

**Write a general formula for finding the slope $m$ of any secant line. Use the formula to approximate the rate of change of $f(x) = 2x^2 - 3x$ for $x = 2$.**

**Step 1** Generate an expression for finding the average rate of change for the figure at the right. This expression is called the *difference quotient*.

**Step 2** Use your difference quotient to approximate the rate of change of $f(x)$ at $x = 2$. Let $h = 0.4$, 0.25, and 0.1.



### Analyze the Results

3. As the value of $h$ gets closer and closer to 0, what does the average rate of change appear to approach? **Sample answer: 5**
4. Graph $f(x)$ and the secant line created when $h = 0.1$. **See margin.**
5. What does the secant line appear to become as $h$ approaches 0?
6. Make a conjecture about the rate of change of a function at a point as it relates to your answer to the previous question.

We can use the difference quotient to find the *exact* rate of change of a function at a single point.

**Activity 3** Calculate Rate of Change

**Use the difference quotient to calculate the *exact* rate of change of $f(x) = 2x^2 - 3x$ at the point $x = 2$.**

**Step 1** Substitute $x = 2$ into the difference quotient, as shown.

$$m = \frac{f(2 + h) - f(2)}{h}$$

**Step 2** Expand the difference quotient by evaluating for $f(2 + h)$ and $f(2)$.

$$m = \frac{[2(2 + h)^2 - 3(2 + h)] - [2(2)^2 - 3(2)]}{h}$$

**Step 3** Simplify the expression. At some point, you will need to factor $h$ from the numerator and then reduce.

**Step 4** Find the *exact* rate of change of $f(x)$ at $x = 2$ by substituting $h = 0$ into your expression.

### Analyze the Results

7. Compare the *exact* rate of change found in Step 4 to the previous rates of change that you found.
8. What happens to the secant line for $f(x)$ at $x = 2$ when $h = 0$?
9. Explain the process for calculating the exact rate of change of a function at a point using the difference quotient.

### Model and Apply

10. In this problem, you will approximate the rate of change for, and calculate the *exact* rate for, $f(x) = x^2 + 1$ at $x = 1$.
    a. Approximate the rate of change of $f(x)$ at $x = 1$ by calculating the average rates of change of the three secant lines through $f(3)$, $f(2)$, and $f(1.5)$. Graph $f(x)$ and the three secant lines on the same coordinate plane. **See margin.**
    b. Approximate the rate of change of $f(x)$ at $x = 1$ by using the difference quotient and three different values for $h$. Let $h = 0.4, 0.25$, and $0.1$. **See margin.**
    c. Calculate the exact rate of change of $f(x)$ at $x = 1$ by first evaluating the difference quotient for $f(1 + h)$ and $f(1)$ and then substituting $h = 0$. Follow the steps in Activity 3. **2**



**5. Sample answer: The secant line appears to become a line tangent to the graph of $f$.**

**6. Sample answer: The rate of change of a function at a point is equivalent to the slope of the line tangent to the graph of the function at that point.**

**7. Sample answer: The exact rate of change is 5 which is equivalent to what the other values appear to be approaching.**

**8. Sample answer: The secant line becomes a line tangent to $f(x)$.**

**9. Sample answer: To find the rate of change of a function $f(x)$ at the point $x$, find the difference quotient, and simplify the expression. Substitute $h = 0$ into the simplified expression, and the result is the rate of change of the function at $x$.**



**Practice** Have students complete Exercises 1–9.

## 3 Assess

### Formative Assessment

Use Model and Apply Exercise 10 to assess students' understanding of how to find the rate of change at a point for nonlinear functions.

### From Concrete to Abstract

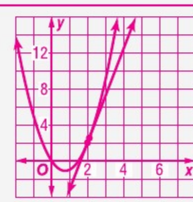
Have students summarize what they have learned about rate of change at a point. Have them include a description of how approximations become more accurate as the points on the graph of the function are closer together.

### Extending the Concept

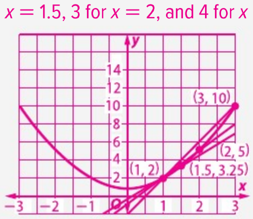
Have students note the changes in rate of change as points approach relative maxima and minima. Students should determine that the rate of change becomes smaller in magnitude as the points of the secant line get closer to the extrema. Have students graph $y = x^3 - 2x + 1$. At what points on the graph will the rates of change be zero? at relative maxima and minima What is the connection between rates of change equal to zero and relative extrema? The rate of change at relative maxima and minima will always be zero.

### Additional Answers

**4.**




**10a.** Sample answer: rate of change: 2.5 for $x = 1.5$, 3 for $x = 2$, and 4 for $x = 3$



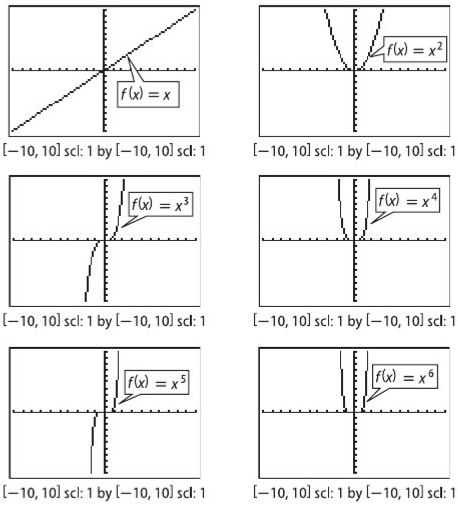

**10b.** For $h = 0.4$, the rate of change is 2.4. For $h = 0.25$, the rate of change is 2.25. For $h = 0.1$, the rate of change is 2.1.

## Lesson 11-1

**77.** No; sample answer: Most nonnegative $x$-values are paired with two $y$-values because it is necessary to take both the positive and negative values of the absolute value of $x$ when solving the equation for $y$.

**78.** Yes; sample answer: Each $x$-value is paired with exactly one $y$-value, so the equation represents a function.

**79a.**




[−10, 10] scl: 1 by [−10, 10] scl: 1 (each graph)

**79b.**

| $n$ | Range |
|---|---|
| 1 | $(-\infty, \infty)$ |
| 2 | $[0, \infty)$ |
| 3 | $(-\infty, \infty)$ |
| 4 | $[0, \infty)$ |
| 5 | $(-\infty, \infty)$ |
| 6 | $[0, \infty)$ |

**79c.** Sample answer: When $n$ is even in $f(x) = x^n$, the range is $[0, \infty)$.

**80.** Tarek; sample answer: The domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ or $\{x \mid x \neq -2, x \neq 2, x \in \mathbb{R}\}$.

**81.** $(-\infty, -3) \cup (-3, -1) \cup (-1, 5) \cup (5, \infty)$; $\{x \mid x \neq -3, x \neq -1, x \neq 5, x \in \mathbb{R}\}$; Sample answer: In this case, set-builder notation is preferential because it is more concise to list the three real numbers to which $x$ cannot be equal rather than the four intervals of which $x$ is an element.

**84.** False; sample answer: Every element in $X$ must be matched with exactly one element in $Y$.

**85.** False; sample answer: Two or more elements in $X$ may be matched with the same element in $Y$.

**87.** Sample answer: If each possible input is assigned to exactly one output in the verbal description, the relation is a function.

**88.** Sample answer: If each $x$-coordinate in the set of ordered pairs is paired with a unique $y$-coordinate, the relation is a function.

**89.** Sample answer: If each input value in the table is paired with a unique output value, the relation is a function.

**90.** Sample answer: If a vertical line drawn at any $x$-value on the graph intersects the graph exactly once, the relation is a function.

**91.** Sample answer: If each $x$-value can be paired with exactly one $y$-value after the equation is solved for $y$, the relation is a function.

## Lesson 11-2

**24.** $x$-axis, $y$-axis, and origin;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| 1 | $\frac{\sqrt{15}}{2}$ | $\left(1, \frac{\sqrt{15}}{2}\right)$ |
| 1 | $-\frac{\sqrt{15}}{2}$ | $\left(1, -\frac{\sqrt{15}}{2}\right)$ |
| 2 | $\sqrt{3}$ | $(2, \sqrt{3})$ |
| 2 | $-\sqrt{3}$ | $(2, -\sqrt{3})$ |
| 3 | $\frac{\sqrt{7}}{2}$ | $\left(3, \frac{\sqrt{7}}{2}\right)$ |
| 3 | $-\frac{\sqrt{7}}{2}$ | $\left(3, -\frac{\sqrt{7}}{2}\right)$ |

Because $x^2 + 4(-y)^2 = 16$ is equivalent to $x^2 + 4y^2 = 16$, the graph is symmetric with respect to the $x$-axis.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| −3 | $\frac{\sqrt{7}}{2}$ | $\left(-3, \frac{\sqrt{7}}{2}\right)$ |
| −2 | $\sqrt{3}$ | $(-2, \sqrt{3})$ |
| −1 | $\frac{\sqrt{15}}{2}$ | $\left(-1, \frac{\sqrt{15}}{2}\right)$ |
| 1 | $\frac{\sqrt{15}}{2}$ | $\left(1, \frac{\sqrt{15}}{2}\right)$ |
| 2 | $\sqrt{3}$ | $(2, \sqrt{3})$ |
| 3 | $\frac{\sqrt{7}}{2}$ | $\left(3, \frac{\sqrt{7}}{2}\right)$ |

Because $(-x)^2 + 4y^2 = 16$ is equivalent to $x^2 + 4y^2 = 16$, the graph is symmetric with respect to the $y$-axis.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| 0 | −2 | $(0, -2)$ |
| −3 | $-\frac{\sqrt{7}}{2}$ | $\left(-3, -\frac{\sqrt{7}}{2}\right)$ |
| −2 | $-\sqrt{3}$ | $(-2, -\sqrt{3})$ |
| 2 | $\sqrt{3}$ | $(2, \sqrt{3})$ |
| 3 | $\frac{\sqrt{7}}{2}$ | $\left(3, \frac{\sqrt{7}}{2}\right)$ |
| 0 | 2 | $(-0, 2)$ |

Because $(-x)^2 + 4(-y^2) = 16$ is equivalent to $x^2 + 4y^2 = 16$, the graph is symmetric with respect to the origin.

**25.** $x$-axis;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| 1 | 2 | (1, 2) |
| 1 | $-2$ | $(1, -2)$ |
| 2 | $\sqrt{5}$ | $(2, \sqrt{5})$ |
| 2 | $-\sqrt{5}$ | $(2, -\sqrt{5})$ |
| 3 | $\sqrt{6}$ | $(3, \sqrt{6})$ |
| 3 | $-\sqrt{6}$ | $(3, -\sqrt{6})$ |

Because $x = (-y)^2 - 3$ is equivalent to $x = y^2 - 3$, the graph is symmetric with respect to the $x$-axis.

**26.** origin;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-4$ | 4 | $(-4, 4)$ |
| $-3$ | 3 | $(-3, 3)$ |
| $-2$ | 2 | $(-2, 2)$ |
| 2 | $-2$ | $(2, -2)$ |
| 3 | $-3$ | $(3, -3)$ |
| 4 | $-4$ | $(4, -4)$ |

Because $-x = -(-y)$ is equivalent to $x = -y$, the graph is symmetric with respect to the origin.

**27.** $x$-axis, $y$-axis, and origin;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| 1 | $\frac{2\sqrt{2}}{5}$ | $\left(1, \frac{2\sqrt{2}}{5}\right)$ |
| 1 | $-\frac{2\sqrt{2}}{5}$ | $\left(1, -\frac{2\sqrt{2}}{5}\right)$ |
| 2 | $\frac{\sqrt{35}}{5}$ | $\left(2, \frac{\sqrt{35}}{5}\right)$ |
| 2 | $-\frac{\sqrt{35}}{5}$ | $\left(2, -\frac{\sqrt{35}}{5}\right)$ |
| 3 | $\frac{4\sqrt{5}}{5}$ | $\left(3, \frac{4\sqrt{5}}{5}\right)$ |
| 3 | $-\frac{4\sqrt{5}}{5}$ | $\left(3, -\frac{4\sqrt{5}}{5}\right)$ |

Because $9x^2 - 25(-y)^2 = 1$ is equivalent to $9x^2 - 25y^2 = 1$, the graph is symmetric with respect to the $x$-axis.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | $\frac{4\sqrt{5}}{5}$ | $\left(-3, \frac{4\sqrt{5}}{5}\right)$ |
| $-2$ | $\frac{\sqrt{35}}{5}$ | $\left(-2, \frac{\sqrt{35}}{5}\right)$ |
| $-1$ | $\frac{2\sqrt{2}}{5}$ | $\left(-1, \frac{2\sqrt{2}}{5}\right)$ |
| 1 | $\frac{2\sqrt{2}}{5}$ | $\left(1, \frac{2\sqrt{2}}{5}\right)$ |
| 2 | $\frac{\sqrt{35}}{5}$ | $\left(2, \frac{\sqrt{35}}{5}\right)$ |
| 3 | $\frac{4\sqrt{5}}{5}$ | $\left(3, \frac{4\sqrt{5}}{5}\right)$ |

Because $9(-x)^2 - 25y^2 = 1$ is equivalent to $9x^2 - 25y^2 = 1$, the graph is symmetric with respect to the $y$-axis.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | $-\frac{4\sqrt{5}}{5}$ | $\left(-3, -\frac{4\sqrt{5}}{5}\right)$ |
| $-2$ | $-\frac{\sqrt{35}}{5}$ | $\left(-2, -\frac{\sqrt{35}}{5}\right)$ |
| $-1$ | $-\frac{2\sqrt{2}}{5}$ | $\left(-1, -\frac{2\sqrt{2}}{5}\right)$ |
| 1 | $\frac{2\sqrt{2}}{5}$ | $\left(1, \frac{2\sqrt{2}}{5}\right)$ |
| 2 | $\frac{\sqrt{35}}{5}$ | $\left(2, \frac{\sqrt{35}}{5}\right)$ |
| 3 | $\frac{4\sqrt{5}}{5}$ | $\left(3, \frac{4\sqrt{5}}{5}\right)$ |

Because $9(-x)^2 - 25(-y)^2 = 1$ is equivalent to $9x^2 - 25y^2 = 1$, the graph is symmetric with respect to the origin.

**28.** origin;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-4$ | $-16$ | $(-4, -16)$ |
| $-2$ | $-2$ | $(-2, -2)$ |
| $-1$ | $-\frac{1}{4}$ | $\left(-1, -\frac{1}{4}\right)$ |
| 1 | $\frac{1}{4}$ | $\left(1, \frac{1}{4}\right)$ |
| 2 | 2 | (2, 2) |
| 4 | 16 | (4, 16) |

Because $-y = \frac{(-x)^3}{4}$ is equivalent to $y = \frac{x^3}{4}$, the graph is symmetric with respect to the origin.

**29.** origin;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-10$ | 1 | $(-10, 1)$ |
| $-5$ | 2 | $(-5, 2)$ |
| $-1$ | 10 | $(-1, 10)$ |
| 1 | $-10$ | $(1, -10)$ |
| 5 | $-2$ | $(5, -2)$ |
| 10 | $-1$ | $(10, -1)$ |

Because $-y = -\frac{10}{(-x)}$ is equivalent to $y = -\frac{10}{x}$, the graph is symmetric with respect to the origin.

**30.** none

**31.** $y$-axis;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | 9 | $(-3, 9)$ |
| $-2$ | $-16$ | $(-2, -16)$ |
| $-1$ | $-7$ | $(-1, -7)$ |
| 1 | $-7$ | $(1, -7)$ |
| 2 | $-16$ | (2, 16) |
| 3 | 9 | (3, 9) |

Because $y = (-x)^4 - 8(-x)^2$ is equivalent to $y = x^4 - 8x^2$, the graph is symmetric with respect to the $y$-axis.

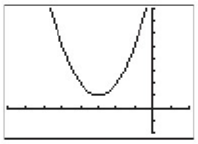
**32.** none

**33.** *y*-axis;

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2\sqrt{2}$ | 6 | $(-2\sqrt{2}, 6)$ |
| $-\frac{\sqrt{14}}{2}$ | 12 | $\left(-\frac{\sqrt{14}}{2}, 12\right)$ |
| $-1$ | $2\sqrt{14}+6$ | $(-1, 2\sqrt{14}+6)$ |
| $\frac{\sqrt{14}}{2}$ | 12 | $\left(\frac{\sqrt{14}}{2}, 12\right)$ |
| 1 | $2\sqrt{14}+6$ | $(1, 2\sqrt{14}+6)$ |
| $2\sqrt{2}$ | 6 | $(2\sqrt{2}, 6)$ |

Because $(y-6)^2 + 8(-x)^2 = 64$ is equivalent to $(y-6)^2 + 8x^2 = 64$, the graph is symmetric with respect to the *y*-axis.

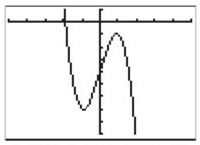
**34.**



[−8, 2] scl: 1 by [−2, 8] scl: 1

neither;
$f(-x) = (-x)^2 + 6(-x) + 10$
$= x^2 - 6x + 10$

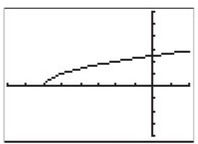
**35.**



[−5, 5] scl: 1 by [−9, 1] scl: 1

neither;
$f(-x) = -2(-x)^3 + 5(-x) - 4$
$= 2x^3 - 5x - 4$

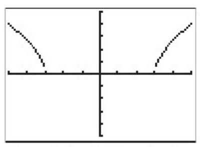
**36.**



[−8, 2] scl: 1 by [−4, 6] scl: 1

neither;
$g(-x) = \sqrt{-x+6}$

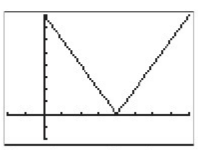
**37.**



[−5, 5] scl: 1 by [−5, 5] scl: 1

Even; the graph of $h(x)$ is symmetric with respect to the *y*-axis.
$h(-x) = \sqrt{(-x)^2 - 9}$
$= \sqrt{x^2 - 9} = h(x)$
$-h(x) = -\sqrt{x^2 - 9}$

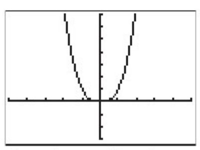
**38.**



[−2, 8] scl: 1 by [−2, 8] scl: 1

neither;
$h(-x) = |8 - 2(-x)|$
$= |8 + 2x|$

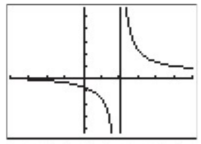
**39.**



[−5, 5] scl: 1 by [−3, 7] scl: 1

Even; the graph of $f(x)$ is symmetric with respect to the *y*-axis.
$f(-x) = |(-x)^3|$
$= |-x^3| = f(x)$
$-f(x) = -|x^3|$

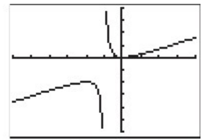
**40.**



[−4, 6] scl: 1 by [−13, 17] scl: 3

neither;
$f(-x) = \frac{-x+4}{-x-2}$

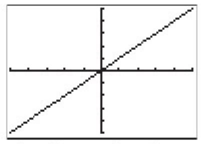
**41.**



[−6, 4] scl: 1 by [−12, 8] scl: 2

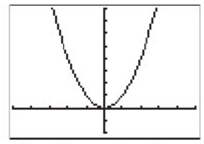
neither;
$g(-x) = \frac{(-x)^2}{-x+1}$
$= \frac{x^2}{-x+1}$

**46a.** $f(x) = x$



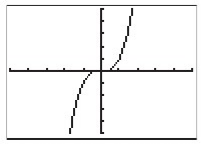
[−5, 5] scl: 1 by [−5, 5] scl: 1

$f(x) = x^2$



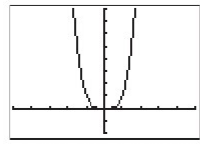
[−5, 5] scl: 1 by [−2, 8] scl: 1

$f(x) = x^3$



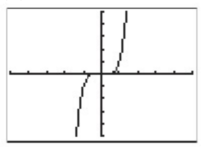
[−5, 5] scl: 1 by [−5, 5] scl: 1

$f(x) = x^4$



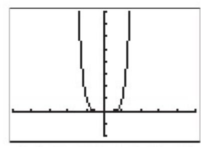
[−5, 5] scl: 1 by [−2, 8] scl: 1

$f(x) = x^5$



[−5, 5] scl: 1 by [−5, 5] scl: 1

$f(x) = x^6$



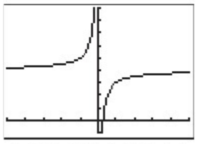
[−5, 5] scl: 1 by [−2, 8] scl: 1

**46b.** $f(x) = x$: D = $(-\infty, \infty)$, R = $(-\infty, \infty)$
$f(x) = x^2$: D = $(-\infty, \infty)$, R = $[0, \infty)$
$f(x) = x^3$: D = $(-\infty, \infty)$, R = $(-\infty, \infty)$
$f(x) = x^4$: D = $(-\infty, \infty)$, R = $[0, \infty)$
$f(x) = x^5$: D = $(-\infty, \infty)$, R = $(-\infty, \infty)$
$f(x) = x^6$: D = $(-\infty, \infty)$, R = $[0, \infty)$

**46c.** $f(x) = x$, $f(x) = x^3$, and $f(x) = x^5$ are symmetric with respect to the origin; $f(x) = x^2$, $f(x) = x^4$, and $f(x) = x^6$ are symmetric with respect to the *y*-axis.

**46d.** D = $(-\infty, \infty)$, R = $(-\infty, \infty)$; $f(x) = x^{35}$ is an odd function, and the graph of the function will be symmetric with respect to the origin.

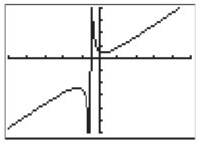
**48.**



[−5, 5] scl: 1 by [−1, 9] scl: 1

$$\frac{4x-1}{x}=0$$
$$\frac{4x}{x}-\frac{1}{x}=0$$
$$4-\frac{1}{x}=0$$
$$-\frac{1}{x}=-4$$
$$\frac{1}{x}=4$$
$$x=\frac{1}{4}$$

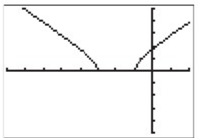
**49.**



[−30, 30] scl: 6 by [−36, 24] scl: 6

no zeros

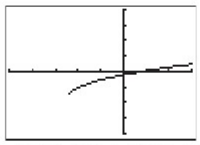
**50.**



[−8, 2] scl: 1 by [−5, 5] scl: 1

$$\sqrt{x^2+4x+3}=0$$
$$x^2+4x+3=0$$
$$(x+3)(x+1)=0$$
$$x+3=0 \text{ or } x+1=0$$
$$x=-3 \quad =-1$$

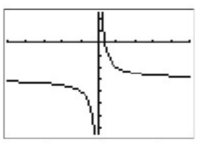
**51.**



[−25, 15] scl: 5 by [−20, 20] scl: 5

$$2\sqrt{x+12}-8=0$$
$$2\sqrt{x+12}=8$$
$$\sqrt{x+12}=4$$
$$x+12=16$$
$$x=4$$

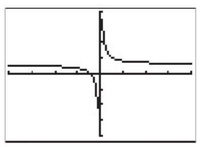
**52.**



[−5, 5] scl: 1 by [−30, 10] scl: 4

$$-12+\frac{4}{x}=0$$
$$\frac{4}{x}=12$$
$$4=12x$$
$$x=\frac{1}{3}$$

**53.**



[−20, 20] scl: 5 by [−20, 20] scl: 5

$$\frac{6}{x}+3=0$$
$$\frac{6}{x}=-3$$
$$-3x=6$$
$$x=-2$$

**59a.** D = [0, 80], R ≈ [−19.19, 12.58]

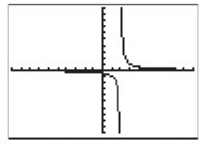
**59b.** Sample answers: 12, 12.58; The *y*-intercept represents the percent change in population in 1930.

**59c.** 15.4 The zeros represent the times at which the percent change in population was 0.

**59d.** 203.83%; Sample answer: This value does not seem realistic because it would mean that the population increased by about 204%, which does not seem likely in comparison to the population trends from 1930 to 2010.

**61b.** Sample answer: As $x$ approaches 2 from the left, the value of $f(x)$ becomes greater and greater. As $x$ approaches 2 from the right, the value of $f(x)$ becomes more and more negative.

**61c.**

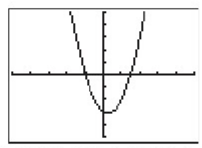


[−10, 10] scl: 1 by [−10, 10] scl: 1

Sample answer: As $x$ approaches 2 from the left, the graph decreases without bound. As $x$ approaches 2 from the right, the graph increases without bound.

**61d.** Sample answer: As the value of $x$ becomes increasingly large, or when $x > 3$, the denominator of the fraction will also become increasingly large. This creates smaller fractions that will continue to decrease but will never reach zero or cross the $x$-axis. This also applies as $x$ decreases at an increasing rate, or when $x < -1$. However, since $x$ is negative, the value will also be negative but it too will never reach 0. As the value of $x$ approaches 2, the difference of $x$ and 2 becomes smaller and smaller. When the difference $d$ is $-1 < d < 1$, the denominator is smaller than the numerator, thus producing a larger number. If the difference is positive, the fraction will approach infinity. If the difference is negative, the fraction will approach negative infinity. Only at 2 will the fraction fail to exist because the difference in the denominator cannot be 0.

**62.**

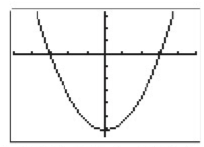


[−10, 10] scl: 2 by [−10, 10] scl: 2

neither;

$$f(-x)=(-x)^2-(-x)-6$$
$$=x^2+x-6$$

**63.**

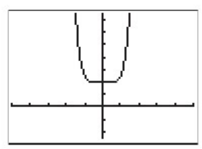


[−10, 10] scl: 2 by [−40, 20] scl: 6

Even; the graph of $g(n)$ is symmetric with respect to the y-axis.

$$g(-n)=n^2-37$$
$$=(-n)^2-37$$
$$=n^2-37=g(n)$$
$$-g(n)=-n^2+37$$

**64.**

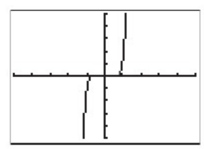


[−5, 5] scl: 1 by [−5, 15] scl: 2

Even; the graph of $h(x)$ is symmetric with respect to the $y$-axis.

$$h(-x)=(-x)^6+4$$
$$=x^6+4=h(x)$$
$$-h(x)=-x^6-4$$

**65.**

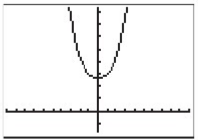


[−5, 5] scl: 1 by [−5, 5] scl: 1

Odd; the graph of $f(g)$ is symmetric with respect to the origin.

$$f(-g)=(-g)^9$$
$$=-g^9=-f(g)$$
$$-f(g)=-g^9$$

**66.**



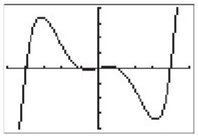
[−10, 10] scl: 1 by [−50, 250] scl: 50

Even; the graph of $g(y)$ is symmetric with respect to the $y$-axis.

$$\begin{aligned} g(-y) &= (-y)^4 + 8(-y)^2 + 81 \\ &= y^4 + 8y^2 + 81 \\ &= g(y) \end{aligned}$$

$-g(y) = -y^4 - 8y^2 - 81$

**67.**



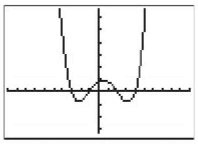
[−5, 5] scl: 1 by [−200, 200] scl: 50

Odd; the graph of $h(y)$ is symmetric with respect to the origin.

$$\begin{aligned} h(-y) &= (-y)^5 - 17(-y)^3 + 16(-y) \\ &= -y^5 + 17y^3 - 16y \\ &= -h(y) \end{aligned}$$

$-h(y) = -y^5 + 17y^3 - 16y$

**68.**



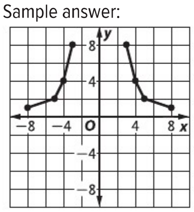
[−10, 10] scl: 1 by [−100, 200] scl: 30

$$\begin{aligned} h(-b) &= (-b)^4 - 2(-b)^3 - 13(-b)^2 + 14(-b) + 24 \\ &= b^4 + 2b^3 - 13b^2 - 14b + 24 \end{aligned}$$

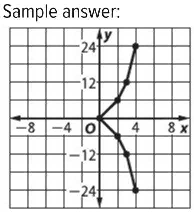
$-h(b) = -b^4 + 2b^3 + 13b^2 - 14b - 24$

The function is neither even nor odd.

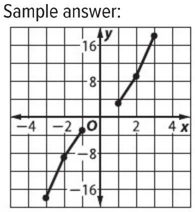
**69.** Sample answer:



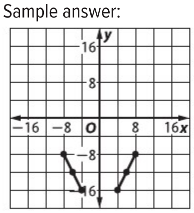

**70.** Sample answer:




**71.** Sample answer:

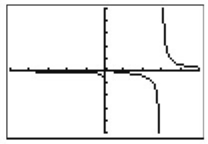



**72.** Sample answer:




**73.** Sample answer: For a relation to be a function, each $x$-value must map to exactly one $y$-value. A relation with two $y$-intercepts located at $(0, y_1)$ and $(0, y_2)$ cannot be a function because two different $y$-values cannot map to the same $x$-value. A relation with two $x$-intercepts located at $(x_1, 0)$ and $(x_2, 0)$ can be a function because two different $x$-values can map to the same $y$-value.

**74.**



[−10, 10] scl: 2 by [−10, 10] scl: 2

Sample answer: From the graph, it appears that $x \neq 0$ and $x \neq 6$, so $D = (-\infty, 0) \cup (0, 6) \cup (6, \infty)$.

$$\begin{aligned} &x^3 - 4x^2 - 12x = 0 \\ &x(x^2 - 4x - 12) = 0 \\ &x(x - 6)(x + 2) = 0 \\ &x = 0 \quad x - 6 = 0 \quad x + 2 = 0 \\ &\qquad\quad\; x = 6 \qquad\; x = -2 \end{aligned}$$

According to the algebraic confirmation, $x = -2$ must also be excluded from the domain. Therefore, $D = (-\infty, -2) \cup (-2, 0) \cup (0, 6) \cup (6, \infty)$.

**75.** False; sample answer: If $n$ is 0, the range is $\{y \mid y = 0\}$. If $n$ is negative, the range is $\{y \mid y \leq 0, y \in \mathbb{R}\}$. The range will only be $\{y \mid y \geq 0, y \in \mathbb{R}\}$ if $n$ is positive.

**76.** False; sample answer: If $n$ is 0, the range is $\{y \mid y = 0\}$.

**77.** False; sample answer: Consider the odd function $y = x^3$ and the point (2, 8) on its graph. Reflecting the point (2, 8) in the line $y = -x$ produces the point (−8, −2), which is not (−2, −8) on the graph of $y = x^3$.

**78.** True; sample answer: For even values of $n$, the function will be rotated multiples of 360°, thus returning the function to its starting position. For odd values of $n$, the function will be rotated multiples of 180° about the origin. This results in a reflection in the $x$-axis. Reflections in the $x$-axis result in $y$-values switching signs, which allows the function to remain even.

**79.** Odd; sample answer: **Proof:**

**Given:** $a$ is an odd function and $b(x) = a(-x)$

**Prove:** $b$ is an odd function

1. $b(x) = a(-x)$ (Given)
2. $a$ is an odd function (Given)
3. $a(-x) = -a(x)$ (Def. odd function)
4. $b(x) = -a(x)$ (Trans. Prop. of Equality using **1** and **3**)
5. $-b(x) = a(x)$ (Div. Prop. of Equality)
6. $a(x) = -b(x)$ (Reflexive Prop. of Equality)
7. $b(-x) = a(-(-x))$ (Substitute $-x$ for $x$ in **1**)
8. $b(-x) = a(x)$ (Substitute $x$ for $-(-x)$.)
9. $b(-x) = -b(x)$ (Trans. Prop. of Equality using **8** and **6**)
10. $b$ is an odd function. (Def. odd function)

**80.** Odd; sample answer: **Proof:**

**Given:** $a$ is an odd function and $b(x) = -a(x)$

**Prove:** $b$ is an odd function

1. $b(x) = -a(x)$ (Given)
2. $a$ is an odd function (Given)
3. $a(-x) = -a(x)$ (Def. odd function)
4. $b(x) = a(-x)$ (Trans. Prop. of Equality using **1** and **3**)
5. $b(-x) = -a(-x)$ (Substitute $-x$ for $x$ in **1**)
6. $-b(-x) = a(-x)$ (Div. Prop. of Equality)
7. $a(-x) = -b(-x)$ (Reflexive Prop. of Equality)
8. $-b(-x) = b(x)$ (Trans. Prop. of Equality using **6** and **4**)

9. $b(-x) = -b(x)$ (Div. Prop. of Equality)
10. b is an odd function. (Def. odd function)

**81.** Even; sample answer: **Proof:**
**Given:** $a$ is an odd function and $b(x) = [a(x)]^2$
**Prove:** $b$ is an even function
**1.** $b(x) = [a(x)]^2$ (Given)
**2.** $a$ is an odd function (Given)
**3.** $a(-x) = -a(x)$ (Def. odd function)
**4.** $b(-x) = [a(-x)]^2$ (Substitute $-x$ for $x$ in **1**)
**5.** $b(-x) = [-a(x)]^2$ (Trans. Prop. of Equality using **4** and **3**)
**6.** $b(-x) = [a(x)]^2$ (Mult. Prop.)
**7.** $b(-x) = b(x)$ (Trans. Prop. of Equality using **6** and **1**)
**8.** b is an even function. (Def. even function)

**82.** Even; sample answer: **Proof:**
**Given:** $a$ is an odd function and $b(x) = a(|x|)$
**Prove:** $b$ is an even function
**1.** $b(x) = a(|x|)$ (Given)
**2.** $a$ is an odd function (Given)
**3.** $a(-x) = -a(x)$ (Def. odd function)
**4.** $b(-x) = a(|-x|)$ (Substitute $-x$ for $x$ in **1**)
**5.** $b(-x) = a(|x|)$ (Def. of absolute value)
**6.** $b(-x) = b(x)$ (Trans. Prop. of Equality using **5** and **1**)
**7.** $b$ is an even function. (Def. even function)

**83.** Odd; sample answer: **Proof:**
**Given:** $a$ is an odd function and $b(x) = [a(x)]^3$
**Prove:** $b$ is an odd function
**1.** $b(x) = [a(x)]^3$ (Given)
**2.** $a$ is an odd function (Given)
**3.** $a(-x) = -a(x)$ (Def. odd function)
**4.** $b(-x) = [a(-x)]^3$ (Substitute $-x$ for $x$ in **1**)
**5.** $b(-x) = [-a(x)]^3$ (Trans. Prop. of Equality using **4** and **3**)
**6.** $b(-x) = -[a(x)]^3$ (Mult. Prop.)
**7.** $-b(-x) = [a(x)]^3$ (Div. Prop. of Equality)
**8.** $-b(-x) = b(x)$ (Trans. Prop. of Equality using **7** and **1**)
**9.** $b(-x) = -b(x)$ (Div. Prop. of Equality)
**10.** $b$ is an odd function. (Def. odd function)

**84.** Sometimes; sample answer: The relation described by $(x - 4)^2 + y^2 = 3$ is symmetric with respect to the line $x = 4$, but is not a function. The relation described by $y = 5$ is symmetric with respect to the line $x = 4$ and is a function.

**85.** Never; sample answer: Any relation symmetric with respect to the line $y = 2$ will have points that lie the same distance vertically above and below the line. Thus, the graph of any relation, other than the line itself, will not pass the vertical line test and will *never* represent a function.

**86.** Sometimes; sample answer: The relation described by $(x - 2)^2 + (y - 2)^2 = 1$ is symmetric with respect to the line $y = x$, but is not a function. The relation described by $y = -x$ is symmetric about the line $y = x$ and is a function.

**87.** Sometimes; sample answer: The relation described by $x^2 + y^2 = 25$ is symmetric with respect to the $x$- and $y$-axes but is not a function. The relation $y = 0$ is symmetric with respect to the $x$- and $y$-axes and is a function.

**88.** Yes; the zero function $f(x) = 0$ is symmetric with respect to the $y$-axis, and therefore even, because $f(-x) = 0$, and thus $f(-x) = f(x)$. The zero function is also symmetric with respect to the origin, and therefore odd, because $f(-x) = 0$, $-f(x) = 0$, and thus $f(-x) = -f(x)$.

## Lesson 11-3

**7.** Discontinuous at $x = 1$; $h(1)$ is undefined and $h(x)$ approaches $-\infty$ as $x$ approaches 1 from the left and $\infty$ as $x$ approaches 1 from the right, so $h(x)$ has an infinite discontinuity at $x = 1$. Discontinuous at $x = 4$; $h(4)$ is undefined and $\lim_{x \to 4} h(x) = \frac{1}{3}$, so $h(x)$ has a removable discontinuity at $x = 4$.

**8.** Discontinuous at $x = 0$; $h(0)$ is undefined and $h(x)$ approaches $-\infty$ as $x$ approaches 0 from both sides, so $h(x)$ has an infinite discontinuity at $x = 0$. Continuous at $x = 6$; $h(6) = 0$, $\lim_{x \to 6} h(x) = 0$, and $\lim_{x \to 6} h(x) = h(6)$.

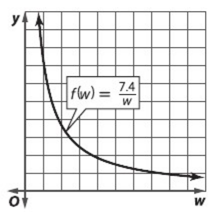
**9.** Discontinuous at $x = -6$; $f(x)$ approaches $-25$ as $x$ approaches $-6$ from the left and 8 as $x$ approaches $-6$ from the right, so $f(x)$ has a jump discontinuity at $x = -6$.

**10.** Discontinuous at $x = -2$; $f(x)$ approaches $-7$ as $x$ approaches $-2$ from the left and 3 as $x$ approaches $-2$ from the right, so $f(x)$ has a jump discontinuity at $x = -2$.

**11a.** Yes; $f(4) = 18.5$, $\lim_{w \to 4} f(w) = 18.5$, and $\lim_{w \to 4} f(w) = f(4)$.

**11b.** Discontinuous; $f(0)$ is undefined and $f(w)$ approaches $\infty$ as $w$ approaches 0 from the right, so $f(w)$ has an infinite discontinuity at $w = 0$.

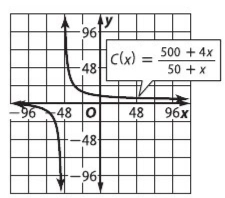
**11c.**




**12a.** Continuous; $C(10) = 9$, $\lim_{x \to 10} C(x) = 9$, and $\lim_{x \to 10} C(x) = C(10)$.

**12b.** Sample answer: No, there is an infinite discontinuity at $x = -50$; $C(-50)$ is undefined and $f(x)$ approaches $-\infty$ as $x$ approaches $-50$ from the left and $\infty$ as $x$ approaches $-50$ from the right. The discontinuity has no affect on the concentration of the mixture because the model is only valid for nonnegative values of $x$.

**12c.**



**22.** From the graph, it appears that $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $4 \times 10^{16}$ | $4 \times 10^{12}$ | 0 | $4 \times 10^{12}$ | $4 \times 10^{16}$ |

**23.** From the graph, it appears that $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $5 \times 10^{12}$ | $5 \times 10^{9}$ | $-1$ | $-5 \times 10^{9}$ | $-5 \times 10^{12}$ |

**24.** From the graph, it appears that $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $-9995$ | $-995$ | $-0.3333$ | 1005 | 10,005 |

**25.** From the graph, it appears that $f(x) \to -4$ as $x \to -\infty$ and $f(x) \to -4$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $-3.9981$ | $-3.9811$ | $-0.8333$ | $-4.0191$ | $-4.0019$ |

**26.** From the graph, it appears that $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $-5000$ | $-500$ | undefined | 499.7 | 4999.7 |

**27.** From the graph, it appears that $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | $-8 \cdot 10^{-4}$ | $-8 \cdot 10^{-3}$ | 0 | $8 \cdot 10^{-3}$ | $8 \cdot 10^{-4}$ |

**28.** From the graph, it appears that $f(x) \to 7$ as $x \to -\infty$ and $f(x) \to 7$ as $x \to \infty$.

| $x$ | $-10{,}000$ | $-1000$ | 0 | 1000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | 7.0000001 | 7.00001 | $-4$ | 7.00001 | 7.0000001 |

**29.** From the graph, it appears that $f(x) \to -2$ as $x \to -\infty$ and $f(x) \to -2$ as $x \to \infty$.

| $x$ | $-1000$ | $-100$ | $-10$ | 0 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1.99999$ | $-1.9999$ | $-1.997$ | $-5$ | $-1.98$ | $-1.9999$ | $-1.99999$ |

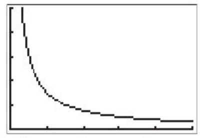
**42.** $g(x)$ has an infinite discontinuity at $x = 1$ because $g(1)$ is undefined and $g(x)$ approaches $\infty$ as $x$ approaches 1 from the left and $-\infty$ as $x$ approaches 1 from the right. From the graph, it appears that as $x \to -\infty$ and $\infty$, $g(x) \to -2$.

| $x$ | $g(x)$ |
|---|---|
| $-10{,}000$ | $-1.9999$ |
| $-1000$ | $-1.999$ |
| 0 | $-1$ |
| 1000 | $-2.001$ |
| 10,000 | $-2.0001$ |

**43.** $h(x)$ has an infinite discontinuity at $x = 0$ because $h(0)$ is undefined and $h(x)$ approaches $\infty$ as $x$ approaches 0 from the left and the right. From the graph, it appears that as $x \to -\infty$ and $\infty$, $h(x) \to 0$.

| $x$ | $h(x)$ |
|---|---|
| $-10{,}000$ | $1.36 \cdot 10^{-8}$ |
| $-1000$ | $1.36 \cdot 10^{-6}$ |
| 0 | undefined |
| 1000 | $1.36 \cdot 10^{-6}$ |
| 10,000 | $1.36 \cdot 10^{-8}$ |

**44a.**



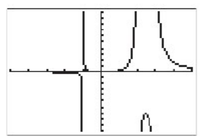
[0, 5] scl: 1 by [0, 10 · 10⁸] scl: 2 · 10⁸

**44b.**

| $\lambda$ | $f(\lambda)$ |
|---|---|
| 0 | undefined |
| 1000 | 299,000 |
| 10,000 | 29,900 |
| 100,000 | 2990 |
| 1,000,000 | 299 |

From the graph, it appears that as $\lambda \to \infty$, $f(\lambda) \to 0$.

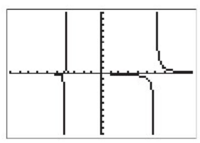
**45.**



[−5, 5] scl: 1 by [−10, 10] scl: 1

discontinuity: infinite at $x = -1$, $x = 2$, and $x = 3$; end behavior: $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to \infty} f(x) = 0$; zero: $x = 0$

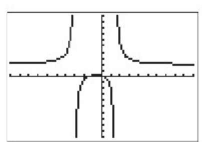
**46.**



[−10, 10] scl: 1 by [−10, 10] scl: 1

discontinuity: infinite at $x = -4$ and $x = 6$, removable at $x = 3$; end behavior: $\lim\limits_{x \to -\infty} g(x) = 0$, $\lim\limits_{x \to \infty} g(x) = 0$; zero: $x = -3$

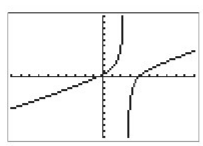
**47.**



[−20, 20] scl: 2 by [−20, 20] scl: 2

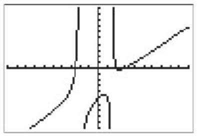
discontinuity: infinite at $x = 3$ and $x = -6$; end behavior: $\lim\limits_{x \to -\infty} h(x) = 4$, $\lim\limits_{x \to \infty} h(x) = 4$; zeros: $x = -3$ and $\frac{1}{4}$

**48.**



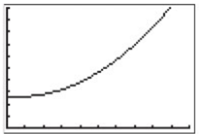
[−20, 20] scl: 2 by [−40, 40] scl: 4

discontinuity: infinite at $x = 5$, removable at $x = -3$; end behavior: $\lim\limits_{x \to -\infty} h(x) = -\infty$, $\lim\limits_{x \to \infty} h(x) = \infty$; zeros: $x = 8$ and $-1$

**49.** 

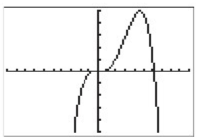
[−20, 20] scl: 2 by [−20, 20] scl: 2

discontinuity: infinite at $x = -4$ and $x = 3$; end behavior: $\lim_{x\to-\infty} h(x) = -\infty$, $\lim_{x\to\infty} h(x) = \infty$; zeros: $x = -5$, 4, and 6

**50a.** 

[0, 20] scl: 2 by [0, 2,000,000] scl: 200,000

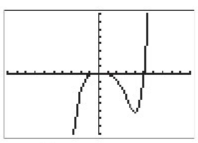
**50c.** Sample answer: From the graph, it appears that as the number of years after 2000 increases without bound, the number of alternative-fueled vehicles increases without bound. Due to money and space constraints, infinitely many alternative-fueled vehicles cannot exist. Therefore, the model is not valid for all years after 2010.

**51.** 

[−20, 20] scl: 2 by [−2500, 2500] scl: 500

end behavior:
$\lim_{x\to-\infty} f(x) = -\infty$,
$\lim_{x\to\infty} f(x) = -\infty$

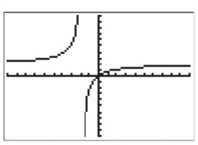
| $x$ | $f(x)$ |
|---|---|
| −10,000 | $-1 \cdot 10^{16}$ |
| −1000 | $-1 \cdot 10^{12}$ |
| −100 | $-1 \cdot 10^{8}$ |
| 0 | −4 |
| 100 | $-9 \cdot 10^{7}$ |
| 1000 | $-1 \cdot 10^{12}$ |
| 10,000 | $-1 \cdot 10^{16}$ |

**52.** 

[−40, 40] scl: 4 by
[−400,000, 400,000] scl: 50,000

end behavior:
$\lim_{x\to-\infty} g(x) = -\infty$,
$\lim_{x\to\infty} g(x) = \infty$

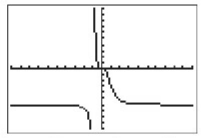
| $x$ | $g(x)$ |
|---|---|
| −10,000 | $-1 \cdot 10^{20}$ |
| −1000 | $-1 \cdot 10^{15}$ |
| −100 | $-1 \cdot 10^{10}$ |
| 0 | −5 |
| 100 | $8 \cdot 10^{9}$ |
| 1000 | $9.8 \cdot 10^{14}$ |
| 10,000 | $1 \cdot 10^{20}$ |

**53.** 

[−80, 80] scl: 8 by [−80, 80] scl: 8

end behavior: $\lim_{x\to-\infty} f(x) = 16$,
$\lim_{x\to\infty} f(x) = 16$

| $x$ | $f(x)$ |
|---|---|
| −10,000 | 16.024 |
| −1000 | 16.244 |
| −100 | 18.824 |
| 0 | – |
| 100 | 13.913 |
| 1000 | 15.764 |
| 10,000 | 15.976 |

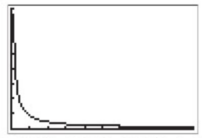
**54.** 

[−20, 20] scl: 2 by [−20, 20] scl: 2

end behavior:
$\lim_{x\to-\infty} g(x) = -12$,
$\lim_{x\to\infty} g(x) = -12$

| $x$ | $g(x)$ |
|---|---|
| −10,000 | −12 |
| −1000 | −12 |
| −100 | −12 |
| 0 | 0 |
| 100 | −12 |
| 1000 | −12 |
| 10,000 | −12 |

**55a.** $f(n) = \frac{3n + 4000}{n}$

**55b.** 

[0, 1000] scl: 100 by [0, 400] scl: 50

**55c.** AED 3; Sample answer: As $n$ approaches $\infty$, $f(n)$ approaches 3. Therefore, the average cost approaches AED 3 as the number of mugs sold increases.

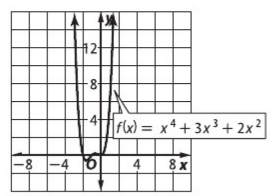
**56a.** $f_1(x) = \frac{3x^3 + 2}{x^3 + 4}$; $f_2(x) = \frac{-x^3 + 5}{x^3 + 7}$; $f_3(x) = \frac{9x^3 - 6}{x^3 + 8}$
$f_1(x) = \frac{6x^3 - 5}{3x^3 + 1}$; $f_2(x) = \frac{3x^3 + 4}{12x^3 + 13}$; $f_3(x) = \frac{7x^3 + 1}{7x^3 - 4}$

**56b.**

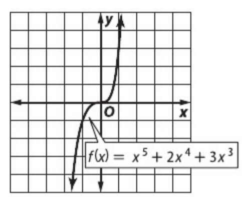
| | $f(x) = \frac{ax^3 + b}{cx^3 + d}$ | | | | | |
|---|---|---|---|---|---|---|
| | $a$ | $b$ | $c$ | $d$ | $\lim_{x\to\infty} f(x)$ | $\lim_{x\to-\infty} f(x)$ |
| $a > c$ | 6 | −5 | 3 | 1 | 2 | 2 |
| $a < c$ | 3 | 4 | 12 | 13 | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $a = c$ | 7 | 1 | 7 | −4 | 1 | 1 |

**56c.** Sample answer: The limit of $f(x) = \frac{ax^3 + b}{cx^3 + d}$ as $x$ approaches $\infty$ or $-\infty$ is $\frac{a}{c}$.

**57a.** Sample answer: If $n$ is even, $f(x)$ approaches $\infty$ as $x$ approaches $-\infty$ and $\infty$.

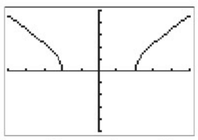



**57b.** Sample answer: If $n$ is odd, $f(x)$ approaches $-\infty$ as $x$ approaches $-\infty$ and $\infty$ as $x$ approaches $\infty$.



**66.** Sample answer: $f(x) = \frac{x(x+3)}{x}$ has a removable discontinuity at $x = 0$. This discontinuity can be eliminated by redefining the function at that point. Let $g(x) = \begin{cases} \frac{x(x+3)}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$. By eliminating the discontinuity, the redefined function $g(x)$ is defined for all $x$-values.

**67.**



[−10, 10] scl: 2 by [−10, 10] scl: 2

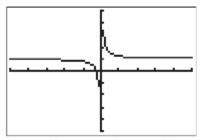
$h(-x) = \sqrt{(-x)^2 - 16}$

$= \sqrt{x^2 - 16} = h(x)$

$-h(x) = -\sqrt{x^2 - 16}$

The function is even, and the graph of the function is symmetric with respect to the $y$-axis.

**68.**



[−10, 10] scl: 2 by [−10, 10] scl: 2

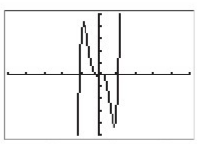
$f(-x) = \frac{2(-x)+1}{-x}$

$= \frac{-2x+1}{-x}$

$-f(x) = \frac{-2x-1}{x} \neq f(-x)$

The function is neither even nor odd.

**69.**



[−10, 10] scl: 2 by [−10, 10] scl: 2

$g(-x) = (-x)^5 - 5(-x)^3 + (-x)$

$= -x^5 + 5x^3 - x = -g(x)$

$-g(x) = -x^5 + 5x^3 - x = -g(-x)$

The function is odd, and the graph of the function is symmetric with respect to the origin.

## Lesson 11-4

**12–21.** Student answers should be close to the approximate extrema values given.

**12.** rel. max: (−0.52, 0.28), (0.68, 0.43); rel. min: (0, 0), (2.24, −9.1)

**13.** abs. max: (−1.41, 3), (1.41, 3); rel. min: (0, −1)

**14.** rel. min: (−2.45, −58.79); rel. max: (2.45, 58.79)

**15.** abs. min: (−3.71, −1334.6); rel. max: (0.11, 0.001); rel. min: (3.59, −1042.52)

**16.** rel. min: (−1.08, −8.58); rel. max: (0.73, 4.29); abs. min: (6.35, −268.71)

**17.** rel. min: (−1, −4); rel. max: (1.08, 4.18)

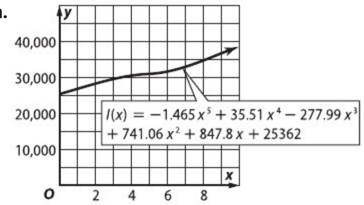
**18.** rel. max: (−1.46, 5.76); rel. min: (1.46, −5.76)

**19.** rel. min: (1.2, −1.11); rel. max: (2, 0)

**20.** rel. max: (−0.95, 4.53); rel. min: (0.91, −3.55); abs. max: (3.79, 22.67)

**21.** rel. max: (−1.14, −1.58); rel. min: (1.14, −8.42)

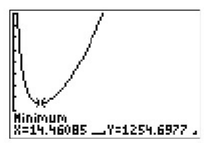
**51a.**




**51b.** about AED 1280.93; Sample answer: The average change in net personal income from 2000 to 2007.

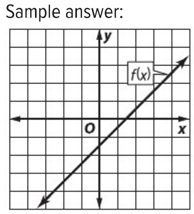
**51c.** The rate of change was lowest from 2000 to 2004 at about AED 826.43 and highest from 2003 to 2007 at about AED 1711.44.

**53.** 14.46 cm × 14.46 cm × 14.46 cm; Sample answer: The function for the surface area of the box in terms of the length of one side of the base is $f(w) = 2w^2 + \frac{12{,}096}{w}$. Graph the function. The absolute minimum occurred when $w \approx 14.46$ cm

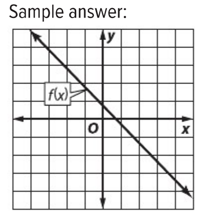



[0, 100] scl: 10 by [0, 5000] scl: 500

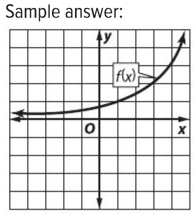
**54.** Sample answer:



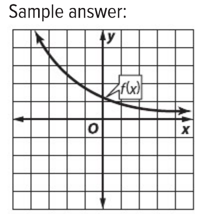

**55.** Sample answer:



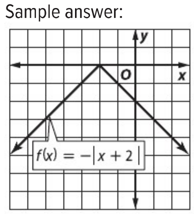

**56.** Sample answer:



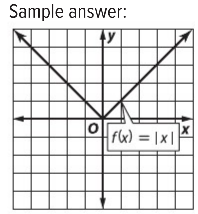

**57.** Sample answer:




**58.** Sample answer:

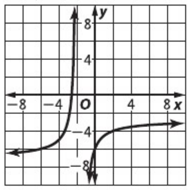



**59.** Sample answer:

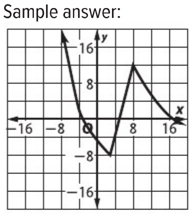


**67.** Sample answer: One reason for the variation in average rate of change may be that Saeed's family encountered traffic later in their trip, thus slowing them down. Another may be that Saeed's family traveled on residential roads for the first hour before entering a highway for the next three hours. The two instances on the graph where Saeed's family appeared to not travel any distance may be as a result of the family needing to stop to eat or take a break. They may have also encountered an accident where traffic was completely stopped.

**69.** Sample answer:



**70.** Sample answer:



**71.** 0; Sample answer: When a function is constant on an interval, the $y$-coordinates of each point in the interval are the same. Because $f(x)$ is constant on the interval $[a, b]$, $f(a) = f(b)$.

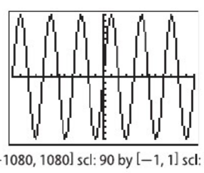
$$m = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{0}{b - a} \text{ or } 0$$

So, the slope of the secant line is zero.

**72.** Sometimes; sample answer: While the average rate of change may be positive, $f(x)$ could be decreasing over a part of the interval and be increasing or constant over other parts of the interval.

**73.**



[−1080, 1080] scl: 90 by [−1, 1] scl: 0.1

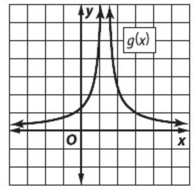
There are an infinite number of relative maxima and minima. The relative maximum is 1 and occurs at $\{x \mid x = 90 + 360n, n \in \mathbb{Z}\}$. The relative minimum is −1 and occurs at $\{x \mid x = 270 + 360n, n \in \mathbb{Z}\}$.

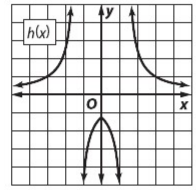
**74.** Sample answer: Because $f(c)$ represents a relative minimum, by definition $f(a)$, where $a$ is slightly less than $c$, must be greater than $f(c)$. Therefore, as $x$ is increasing from $a$ to $c$, the function is decreasing.

**75.** Sample answer: When a function is increasing on an interval, the average rate of change is positive. When a function is decreasing on an interval, the average rate of change is negative. When a function is constant on an interval, the average rate of change is zero.

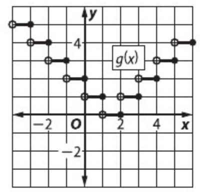
## Lesson 11-5 (Guided Practice)

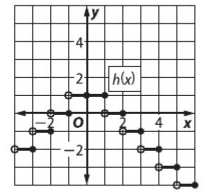
**7A.**



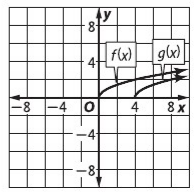

**7B.**



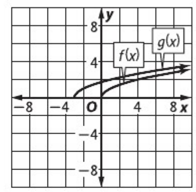




## Lesson 11-5

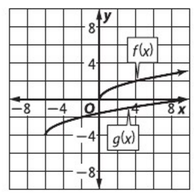
**7.**



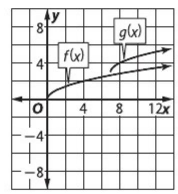

**8.**



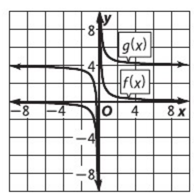

**9.**



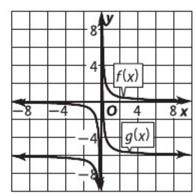

**10.**



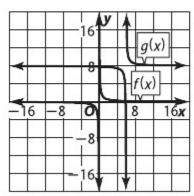

**11.**



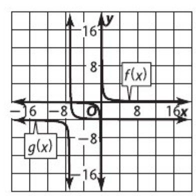

**12.**




**13.**




**14.**




**15.** The graph of $g(x)$ is the graph of $f(x)$ translated 5 units to the right when $g(x) = [\![x - 5]\!]$, or translated 5 units down when $g(x) = [\![x]\!] - 5$.

**16.** The graph of $g(x)$ is the graph of $f(x)$ translated 3 units to the left when $g(x) = [\![x + 3]\!]$, or translated 3 units up when $g(x) = [\![x]\!] + 3$.

**17.** The graph of $g(x)$ is the graph of $f(x)$ reflected in the $y$-axis and translated 5 units right when $g(x) = [\![5 - x]\!]$, or reflected in the $y$-axis and translated 5 units up when $g(x) = [\![-x]\!] + 5$.

**18.** The graph of $g(x)$ is the graph of $f(x)$ reflected in the $y$-axis and translated 2 units to the left when $g(x) = [\![-x - 2]\!]$, or reflected in the $y$-axis and translated 2 units down when $g(x) = [\![-x]\!] - 2$.

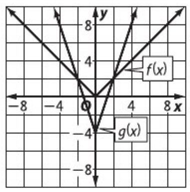
**20.** The graph of $g(x)$ is the graph of $f(x)$ translated 8 units to the right; $g(x) = |x - 8|$.

**21.** The graph of $g(x)$ is the graph of $f(x)$ translated 5 units down; $g(x) = |x| - 5$.

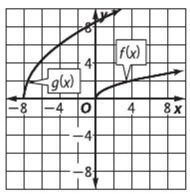
**22.** The graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the left and 8 units down; $g(x) = |x + 4| - 8$.

**23.** The graph of $g(x)$ is the graph of $f(x)$ translated 1 unit to the right and 2 units down; $g(x) = |x - 1| - 2$.

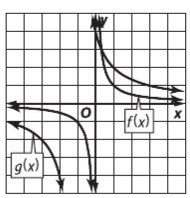
**24.** $f(x) = |x|$; the graph of $g(x)$ is the graph of $f(x)$ expanded vertically and translated 4 units down.



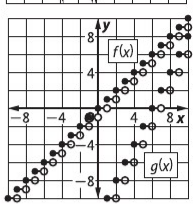

**25.** $f(x) = \sqrt{x}$; the graph of $g(x)$ is the graph of $f(x)$ translated 8 units to the left and expanded vertically.



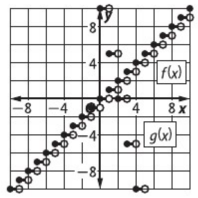

**26.** $f(x) = \frac{1}{x}$; the graph of $g(x)$ is the graph of $f(x)$ translated 1 unit to the left and expanded vertically.



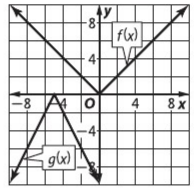

**27.** $f(x) = [\![x]\!]$; the graph of $g(x)$ is the graph of $f(x)$ translated 6 units to the right and expanded vertically.



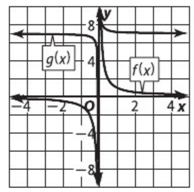

**28.** $f(x) = [\![x]\!]$; the graph of $g(x)$ is the graph of $f(x)$ translated 2 units to the right, expanded vertically, and reflected in the $x$-axis.



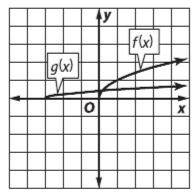

**29.** $f(x) = |x|$; the graph of $g(x)$ is the graph of $f(x)$ translated 5 units to the left, expanded vertically, and reflected in the $x$-axis.



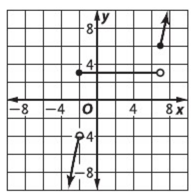

**30.** $f(x) = \frac{1}{x}$; the graph of $g(x)$ is the graph of $f(x)$ compressed vertically and translated 7 units up.



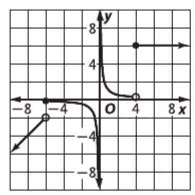

**31.** $f(x) = \sqrt{x}$; the graph of $g(x)$ is the graph of $f(x)$ translated 3 units to the left and compressed vertically.



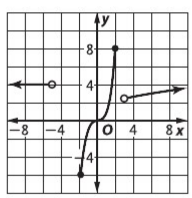

**32.**



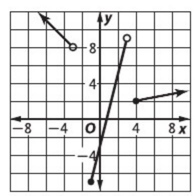
**33.**



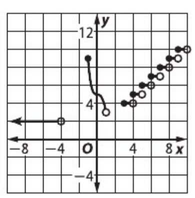
**34.**



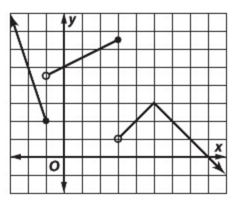
**35.**

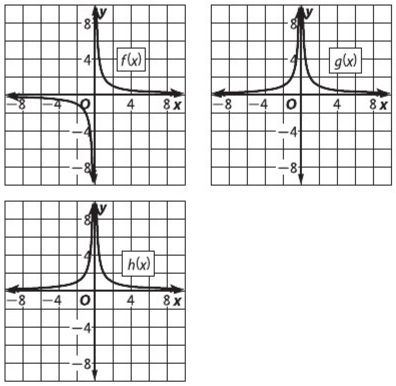


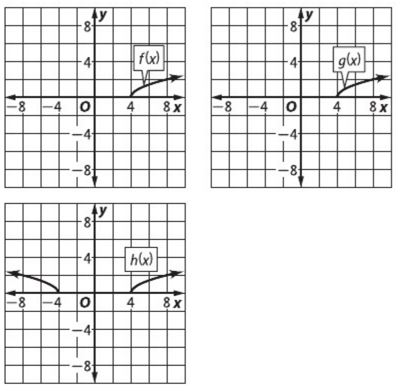
**36.**

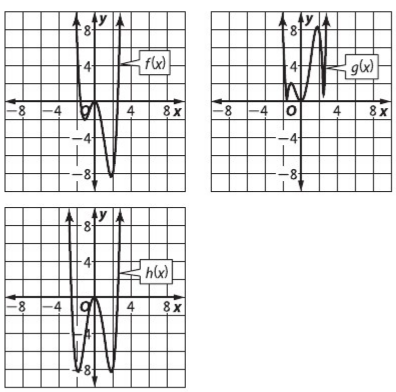


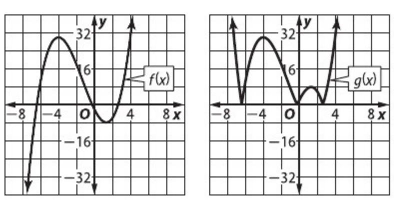
**37.**

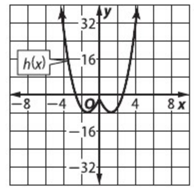


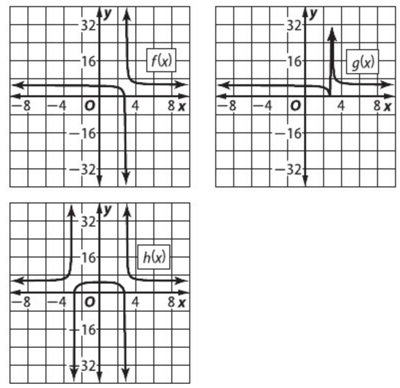
41.


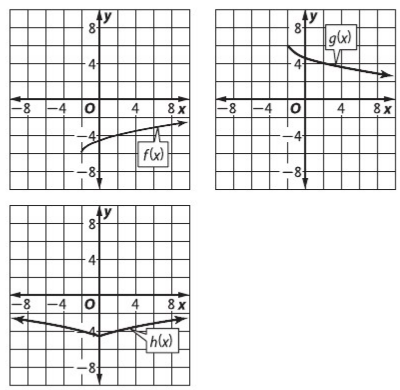

42.



43.


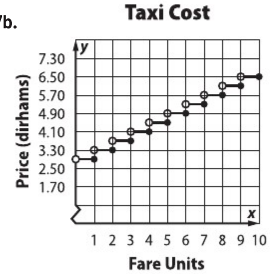

44.


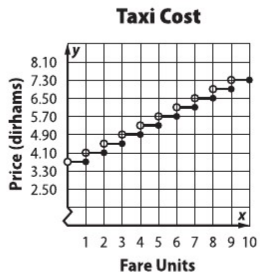

45.



46.


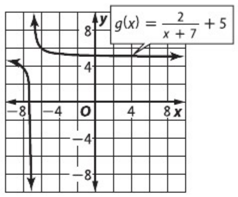

47a. $f(x) = \begin{cases} 0.4\,[\![x]\!] + 2.50 & \text{if } [\![x]\!] = x \\ 0.4\,[\![x+1]\!] + 2.50 & \text{if } [\![x]\!] < x \end{cases}$

47b. **Taxi Cost**


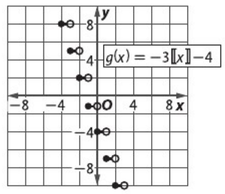
**47c.** The graph of $f(x)$ is translated 0.8 unit up.




**49.** $g(x) = \frac{2}{x+7} + 5$




**50.** $g(x) = -3[\![x]\!] - 4$




**51.** translated one unit to the left; translated one unit down

**52.** translated 10 units up

**53.** translated 2 units to the left; expanded vertically, translated 7 units down

**54.** translated $\frac{5}{3}$ units to the left; expanded vertically; translated $\frac{7}{6}$ units down

**59b.** $g(x) = f(x)$. While the opening is delayed, the number of days after the opening, which determines the domain of the function, is unaffected.

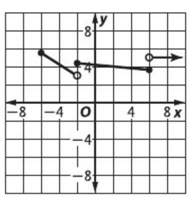
**60.** $f(x) = x^2$; Sample answer: The graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the left, compressed vertically, reflected in the $x$-axis, and translated 3 units down.

**61.** $f(x) = x^3$; Sample answer: The graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the right, expanded vertically, reflected in the $x$-axis, and translated 2 units up.

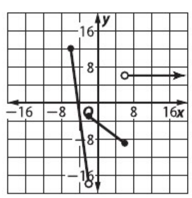
**62.** $f(x) = \frac{1}{x}$; Sample answer: The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and translated 6 units down.

**63.** $f(x) = \sqrt{x}$; Sample answer: The graph of $g(x)$ is the graph of $f(x)$ translated 3 units to the right, reflected in the $x$-axis, and translated 5 units up.

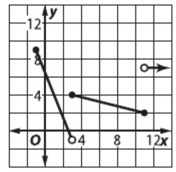
**64.**



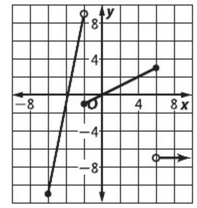
**65.**



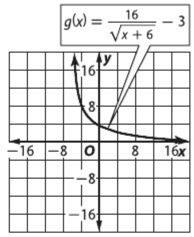
**66.**



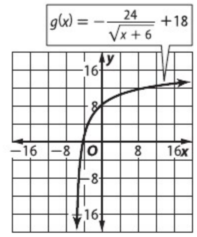
**67.**



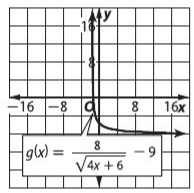
**68.** $g(x) = \frac{16}{\sqrt{x+6}} - 3$



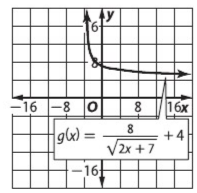

**69.** $g(x) = -\frac{24}{\sqrt{x+6}} + 18$




**70.** $g(x) = \frac{8}{\sqrt{4x+6}} - 9$




**71.** $g(x) = \frac{8}{\sqrt{2x+7}} + 4$




**74.** $f(x)$ and $g(x)$ represent the same function; sample answer: If $f(x) = x^3$, an odd function, then $g(x) = -x^3$, a reflection of $f(x)$ in the $x$-axis. Likewise, $h(x) = -(-x^3)$ or $x^3$, a reflection of $g(x)$ in the $y$-axis. Therefore, $f(x) = h(x)$.

**75.** Sample answer: Order is important because different graphs can be obtained depending on the order the transformations are performed. For example, if $(a, b)$ is on the original graph and there is a translation 6 units up and then a reflection in the $x$-axis, the resulting point will be $(a, -b - 6)$. However, if $(a, b)$ is reflected in the $x$-axis first and then translated 6 units up, the resulting point will be $(a, -b + 6)$.

**76.** Sometimes; sample answer: If $f(x)$ is even, and all values of $f(x)$ are nonnegative, then $f(x) = |f(x)|$. However, $g(x) = x^2 - 4$ is even, but $g(-1) \neq |g(-1)|$.

**77.** Sometimes; sample answer: $f(x) = x^3$ is an odd function and $f(-x) \neq -|f(x)|$ for all values of $x$. However, $f(x) = 0$ is an odd function and $f(-x) = -|f(x)|$ for all $x$.

**78.** Sometimes; sample answer: $f(x) = -x^2$ is an even function and $f(x) = -|f(x)|$ for all $x$. However, $f(x) = x^2$ is an even function and $f(x) \neq -|f(x)|$ when $x = 0$.

**79.** Sample answer: The graph of $g(x) = \sqrt{x+2} - 6$ is the graph of $f(x) = \sqrt{x}$ translated 2 units to the left and 6 units down.

**80.** Sample answer: A vertical expansion of $f(x)$ by a factor of 4 would move $(a, b)$ to $(a, 4b)$. A horizontal compression by a factor of 4 would move $(a, b)$ to $\left(\frac{1}{4}a, b\right)$.

## Lesson 11-6

**5.** $(f+g)(x) = x^2 + 10x$; $D = (-\infty, \infty)$; $(f-g)(x) = x^2 - 8x$; $D = (-\infty, \infty)$; $(f \cdot g)(x) = 9x^3 + 9x^2$; $D = (-\infty, 0) \cup (0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x+1}{9}$; $D = (-\infty, 0) \cup (0, \infty)$

**6.** $(f+g)(x) = 2x$; $D = (-\infty, \infty)$; $(f-g)(x) = -14$; $D = (-\infty, \infty)$; $(f \cdot g)(x) = x^2 - 49$; $D = (-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x-7}{x+7}$; $D = (-\infty, -7) \cup (-7, \infty)$

**7.** $(f+g)(x) = x^3 + x + \frac{6}{x}$; $D = (-\infty, 0) \cup (0, \infty)$; $(f-g)(x) = -x^3 - x + \frac{6}{x}$; $D = (-\infty, 0) \cup (0, \infty)$; $(f \cdot g)(x) = 6x^2 + 6$; $D = (-\infty, 0) \cup (0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{6}{x^4 + x^2}$; $D = (-\infty, 0) \cup (0, \infty)$

**8.** $(f+g)(x) = \frac{x^2+12}{4x}$; $D = (-\infty, 0) \cup (0, \infty)$; $(f-g)(x) = \frac{x^2-12}{4x}$; $D = (-\infty, 0) \cup (0, \infty)$; $(f \cdot g)(x) = \frac{3}{4}$; $D = (-\infty, 0) \cup (0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{x^2}{12}$; $D = (-\infty, 0) \cup (0, \infty)$

**9.** $(f+g)(x) = \frac{1}{\sqrt{x}} + 4\sqrt{x}$; $D = (0, \infty)$; $(f-g)(x) = \frac{1}{\sqrt{x}} - 4\sqrt{x}$; $D = (0, \infty)$; $(f \cdot g)(x) = 4$; $D = (0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{1}{4x}$; $D = (0, \infty)$

**10.** $(f+g)(x) = \frac{3}{x} + x^4$; $D = (-\infty, 0) \cup (0, \infty)$; $(f-g)(x) = \frac{3}{x} - x^4$; $D = (-\infty, 0) \cup (0, \infty)$; $(f \cdot g)(x) = 3x^3$; $D = (-\infty, 0) \cup (0, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{3}{x^5}$; $D = (-\infty, 0) \cup (0, \infty)$

**11.** $(f+g)(x) = \sqrt{x+8} + \sqrt{x+5} - 3$; $D = [-5, \infty)$; $(f-g)(x) = \sqrt{x+8} - \sqrt{x+5} + 3$; $D = [-5, \infty)$; $(f \cdot g)(x) = \sqrt{x^2 + 13x + 40} - 3\sqrt{x+8}$; $D = [-5, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x^2+13x+40} + 3\sqrt{x+8}}{x-4}$; $D = [-5, 4) \cup (4, \infty)$

**12.** $(f+g)(x) = \sqrt{x+6} + \sqrt{x-4}$; $D = [4, \infty)$; $(f-g)(x) = \sqrt{x+6} - \sqrt{x-4}$; $D = [4, \infty)$; $(f \cdot g)(x) = \sqrt{x^2 + 2x - 24}$; $D = [4, \infty)$; $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x^2+2x-24}}{x-4}$; $D = (4, \infty)$

**13a.** $(f+g)(x) = 40x + 550$; $x \geq 0$

**13b.** $(f+g)(x)$ represents all factors that influence the budget.

**13c.** AED 710; The budget for 4 weeks.

**15.** $[f \circ g](x) = 8x - 19$; $[g \circ f](x) = 8x - 20$; $[f \circ g](6) = 29$

**16.** $[f \circ g](x) = -50x^2 + 145x - 101$; $[g \circ f](x) = 10x^2 + 25x + 1$; $[f \circ g](6) = -1031$

**17.** $[f \circ g](x) = -x^4 - 2x^3 - 3x^2 - 2x + 7$; $[g \circ f](x) = x^4 - 17x^2 + 73$; $[f \circ g](6) = -1841$

**18.** $[f \circ g](x) = x^4 + 14x^3 + 71x^2 + 154x + 105$; $[g \circ f](x) = x^4 - 25x^2 + 155$; $[f \circ g](6) = 7905$

**19.** $[f \circ g](x) = -x^6 - 2x^3 + 2$; $[g \circ f](x) = -x^6 + 9x^4 - 27x^2 + 28$; $[f \circ g](6) = -47{,}086$

**20.** $[f \circ g](x) = 2 + x^8$; $[g \circ f](x) = -x^8 - 4x^4 - 4$; $[f \circ g](6) = 1{,}679{,}618$

**21.** $[f \circ g](x) = \frac{1}{x^2 - 3}$ for $x \neq \pm\sqrt{3}$

**22.** $[f \circ g](x) = \frac{2}{x^2 + 3}$

**23.** $[f \circ g](x) = |x|$

**24.** $[f \circ g](x) = x - 6$ for $x \geq -3$

**25.** $[f \circ g](x) = \frac{5\sqrt{6-x}}{6-x}$ for $x < 6$

**26.** $[f \circ g](x) = \frac{-4\sqrt{x+8}}{x+8}$ for $x > -8$

**27.** $[f \circ g](x) = |x + 2|$

**28.** $[f \circ g](x) = \sqrt{x^2 + 6}$

**29a.** $\{v \mid 0 \leq v < c, v \in \mathbb{R}\}$; The speed of the object $v$ cannot be equal to the speed of light $c$. Otherwise, we would obtain $\frac{100}{0}$, which is undefined. Also, the speed $v$ cannot be greater than $c$. Otherwise, we would have to find the square root of a negative number, which is an imaginary number. Lastly, the speed cannot be less than zero because speed cannot be negative.

**29b.** $m(10) = 100$ kg; $m(10{,}000) \approx 100.0000001$ kg; $m(1{,}000{,}000) \approx 100.0005556$ kg

**29c.** As $v$ approaches $c$, $m(v)$ approaches $\infty$.

**29d.** Sample answer: $m(v) = f[g(v)]$; $f(v) = \frac{100}{\sqrt{v}}$; $g(v) = 1 - \frac{v^2}{c^2}$

**30.** Sample answer: $f(x) = \sqrt{x} + 7$; $g(x) = 4x + 2$

**31.** Sample answer: $f(x) = \frac{6}{x} - 8$; $g(x) = x + 5$

**32.** Sample answer: $f(x) = |x| - 9$; $g(x) = 4x + 8$

**33.** Sample answer: $f(x) = [\![-3x]\!]$; $g(x) = x - 9$

**34.** Sample answer: $f(x) = \sqrt{x}$; $g(x) = \frac{5-x}{x+2}$

**35.** Sample answer: $f(x) = x^3$; $g(x) = \sqrt{x} + 4$

**36.** Sample answer: $f(x) = \frac{6}{x^2}$; $g(x) = x + 2$

**37.** Sample answer: $f(x) = \frac{8}{x^2}$; $g(x) = x - 5$

**38.** Sample answer: $f(x) = \frac{\sqrt{x}}{x-6}$; $g(x) = x + 4$

**39.** Sample answer: $f(x) = \frac{x+6}{\sqrt{x}}$; $g(x) = x - 1$

**40b.** The value of $v$ must be greater than 0. The speed of the object cannot be equal to zero or negative. If it is, then wavelength is negative or the function is undefined.

**40d.** Sample answer: $a(v) = \frac{h}{v}$, $b(v) = 25v$, and $f(v) = a[b(v)] = \frac{h}{25v}$

**53.** $[f \circ g \circ h](x) = x + 6\sqrt{x} + 11$ for $x \geq 0$

**54.** $[f \circ g \circ h](x) = 25x - 80\sqrt{x} + 62$ for $x \geq 0$

**55.** $[f \circ g \circ h](x) = \sqrt{\frac{1}{x^2} + 2}$ for $x \neq 0$

**56.** $[f \circ g \circ h](x) = \frac{3}{x^2 - 3}$ for $x \neq \pm\sqrt{3}$

**72.** $[f \circ g](x) = x$ for $x \geq -4$; $[g \circ f](x) = |x - 3| + 3$

**73.** $[f \circ g](x) = x$ for $x \geq -19$; $[g \circ f](x) = |x + 4| - 4$

**74.** $[f \circ g](x) = \sqrt{\sqrt{16 + x^2} + 6}$; $[g \circ f](x) = \sqrt{x + 22}$ for $x \geq -6$

**75.** $[f \circ g](x) = \sqrt{\sqrt{9 - x^2}}$ for $-3 \leq x \leq 3$; $[g \circ f](x) = \sqrt{9 - x}$ for $0 \leq x \leq 9$

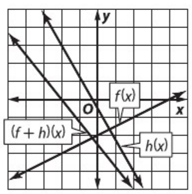
**76.** $[f \circ g](x) = \frac{-8x - 24}{5x + 7}$ for $x \neq -\frac{7}{5}$ and $x \neq -3$;

$[g \circ f](x) = \frac{-8x + 10}{-12x + 7}$ for $x \neq \frac{7}{12}$ and $x \neq \frac{5}{4}$

**77.** $[f \circ g](x) = \frac{24 - 6x}{12 - x}$ for $x \neq 4$ and $x \neq 12$;

$[g \circ f](x) = \frac{4x + 2}{4x - 1}$ for $x \neq -\frac{1}{2}$ and $x \neq \frac{1}{4}$

**78.**

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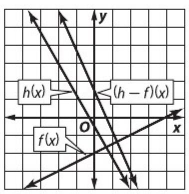


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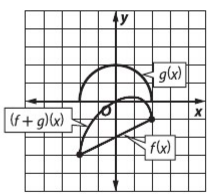


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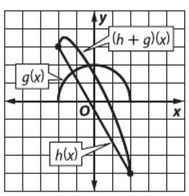
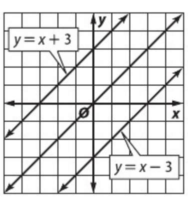
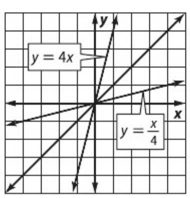
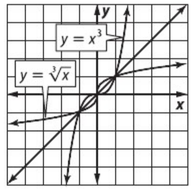



**82b.** Sample answer: The composition of $f$ with $g$ is the same as the composition of $g$ with $f$.

**82c.**




**82d.** Sample answer: The line of reflection between each pair of graphs is the line $y = x$.

**82e.** Sample answer: The compositions $[f \circ g](x)$ and $[g \circ f](x)$ are equivalent to the identity function for each pair of functions.

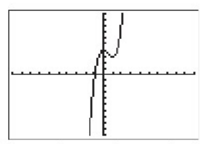
**93.** Sample answer: This can occur when the range of $g(x)$ is a subset of the domain of $f(x)$. For example, when $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$, the range of $g(x)$ is all real numbers greater than or equal to 4. This range is a subset of the domain of $f(x)$, which is all real numbers greater than or equal to 0. Thus, $f[g(x)]$ exists for all real numbers.

**94.** Sample answer: false; counterexample: If $g(x) = x^2 + x + 1$ and $f(x) = \sqrt{x}$, then $[f \circ g](x) = \sqrt{x^2 + x + 1}$, which is not linear.

**96.** Sample answer: First, $g(x)$ must be defined, so $x \neq 3$. Second, $f(x)$ must be defined for $g(x)$, so $g(x)$ must be $\geq 1$, which is true when $3 < x \leq 4$. Finally, the composition must be found and evaluated for additional domain restrictions. The composition is $\sqrt{\frac{4 - x}{x - 3}}$, and there are no additional domain restrictions. The domain of the composition is $\{x \mid 3 < x \leq 4, x \in \mathbb{R}\}$.

**97.** Sample answer: Order is important because there usually will be different outputs for $f[g(x)]$ and $g[f(x)]$. For example, if $f(x) = x^2$ and $g(x) = x + 2$, $f[g(x)] = x^2 + 4x + 4$ and $g[f(x)] = x^2 + 2$.

**99.**



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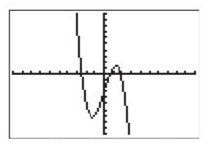
**100.**



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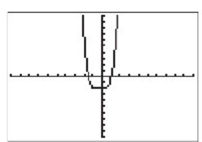
**101.**



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**108a.** Each $x$-value is related to exactly one $y$-value, so the graph passes the Vertical Line Test.

**108d.** There are zeros at approximately 0, 3.2, and 6.3. There is no symmetry for the function and the function does not appear to be even or odd with this domain.

**108e.** The function is continuous at $x = 2$. $f(x)$ is defined at 2. That is, $f(2)$ exists. $f(x)$ approaches the same function value from either side of 2. That is, $\lim_{x \to 2} f(x)$ exists. The function value that $f(x)$ approaches from each side of 2 is $f(2)$. That is, $\lim_{x \to 2} f(x) = f(2)$.

**108h.** When the function is increasing, the substance is warming up, and when the function is decreasing, the substance is cooling down. The substance cooled at an average of 12.3°C per unit of time from $t = 2$ to $t = 5$.

## Lesson 11-7

**27.** $f[g(x)] = \frac{-6(3-x)}{6} + 3 = \frac{-18+6x}{6} + 3 = -3 + x + 3 = x;$

$g[f(x)] = \frac{3-(-6x+3)}{6} = \frac{6x}{6} = x$

**28.** $f[g(x)] = \frac{4(x-9)}{4} + 9 = x - 9 + 9 = x;$

$g[f(x)] = \frac{4x+9-9}{4} = \frac{4x}{4} = x$

**29.** $f[g(x)] = -3\left(\sqrt{\frac{5-x}{3}}\right)^2 + 5 = \frac{-3(5-x)}{3} + 5 = x - 5 + 5 = x;$

$g[f(x)] = \sqrt{\frac{5-(-3x^2+5)}{3}} = \sqrt{\frac{3x^2}{3}} = \sqrt{x^2} = x$

**30.** $f[g(x)] = \frac{(\sqrt{4x-32})^2}{4} + 8 = \frac{4x-32}{4} + 8 = x - 8 + 8 = x;$

$g[f(x)] = \sqrt{4\left(\frac{x^2}{4}+8\right) - 32} = \sqrt{x^2+32-32} = \sqrt{x^2} = x$

**31.** $f[g(x)] = 2\left(\sqrt[3]{\frac{x+6}{2}}\right)^3 - 6 = \frac{2(x+6)}{2} - 6 = x + 6 - 6 = x;$

$g[f(x)] = \sqrt[3]{\frac{2x^3-6+6}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$

**32.** $f[g(x)] = \left(x^{\frac{2}{3}} - 8 + 8\right)^{\frac{3}{2}} = \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = x;$

$g[f(x)] = \left[(x+8)^{\frac{3}{2}}\right]^{\frac{2}{3}} - 8 = x + 8 - 8 = x$

**33.** $f[g(x)] = \left(\sqrt{x+8} - 4\right)^2 + 8\sqrt{x+8} - 32 + 8 = x + 8 - 8\sqrt{x+8} + 16 + 8\sqrt{x+8} - 24 = x + 24 - 24 = x;$

$g[f(x)] = \sqrt{x^2+8x+8+8} - 4 = \sqrt{x^2+8x+16} - 4 = \sqrt{(x+4)^2} - 4 = x + 4 - 4 = x$

**34.** $f[g(x)] = \left(\sqrt{x-8}+5\right)^2 - 10\sqrt{x-8} - 50 + 33 = x - 8 + 10\sqrt{x-8} + 25 - 10\sqrt{x-8} - 17 = x + 17 - 17 = x;$

$g[f(x)] = \sqrt{x^2-10x+33-8} + 5 = \sqrt{x^2-10x+25} + 5 = \sqrt{(x-5)^2} + 5 = x - 5 + 5 = x$

**35.** $f[g(x)] = \frac{\frac{4}{x-1}+4}{\frac{4}{x-1}} = \frac{4+4x-4}{x-1} \div \frac{4}{x-1} = \frac{4+4x-4}{4}$

$= \frac{4x}{4} = x;\ g[f(x)] = \frac{4}{\frac{x+4}{x} - 1} = \frac{4}{\frac{x+4-x}{x}} = 4 \div \frac{4}{x} = x$

**36.** $f[g(x)] = \frac{\frac{2x+6}{1-x} - 6}{\frac{2x+6}{1-x}+2} = \frac{\frac{2x+6-6+6x}{1-x}}{\frac{2x+6+2-2x}{1-x}}$

$= \frac{2x+6-6+6x}{2x+6+2-2x} = \frac{8x}{8} = x;\ g[f(x)] = \frac{\frac{2(x-6)}{x+2}+6}{1-\frac{x-6}{x+2}} =$

$\frac{\frac{2x-12+6(x+2)}{x+2}}{\frac{x+2-(x-6)}{x+2}} = \frac{2x-12+6x+12}{x+2-x+6} = \frac{8x}{8} = x$

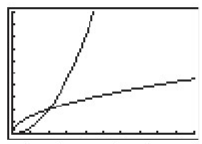
**37a.** $g(x) = \sqrt{\frac{2x}{m}}$; $g(x)$ = velocity in m/s, $x$ = kinetic energy in joules, $m$ = mass in kg

**37b.** $f[g(x)] = f\left(\sqrt{\frac{2x}{m}}\right) = \frac{1}{2}m\left(\sqrt{\frac{2x}{m}}\right)^2 = \frac{1}{2}m\left(\frac{2x}{m}\right) = x;$

$g[f(x)] = g\left(\frac{1}{2}mx^2\right) = \left(\sqrt{\frac{2\left(\frac{1}{2}mx^2\right)}{m}}\right) = \sqrt{x^2} = x;$

because $f[g(x)] = g[f(x)] = x$, the functions are inverses when the domain of $f(x)$ is restricted to $(0, \infty)$.

**37c.**



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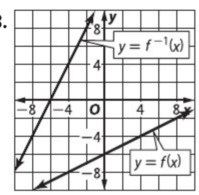
**38.**

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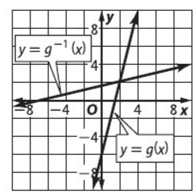


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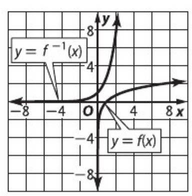


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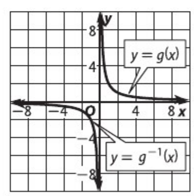


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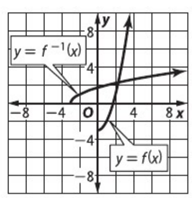


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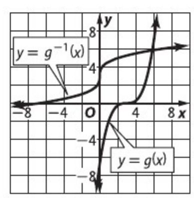



**44a.** Sample answer: The graph of the function is linear, so it passes the horizontal line test. Therefore, it is a one-to-one function and it has an inverse; $f^{-1}(x) = 10x - 5600$.

**44b.** $x$ represents Suha's earnings for a week, and $f^{-1}(x)$ represents her sales.

**44c.** $x \geq 0$; Suha cannot have negative sales.

**63a.** Sample answer: $f(x) = 252.81x + 1757$

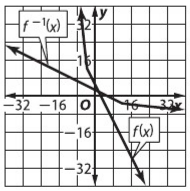
**63b.** $f^{-1}(x) = \frac{x - 1757}{252.81}$; $x$ represents the number of nesting pairs and $f^{-1}(x)$ represents the number of years after 1984.

**64a.** Let $x$ represent the number of stems of hydrangea; $c(x) = 3.5x + 5(75 - x)$.

**64b.** $c^{-1}(x) = 250 - \frac{x}{1.5}$; $x$ represents the total cost and $c^{-1}(x)$ represents the number of stems of hydrangea

**64c.** Domain of $c(x)$: $\{x \mid 0 \leq x \leq 75, x \in \mathbb{W}\}$
Domain of $c^{-1}(x)$: $\{x \mid 262.5 \leq x \leq 375, x \in \mathbb{R}\}$

**65.** $f^{-1}(x) = \begin{cases} -\sqrt{x} & \text{if } 16 \leq x \\ 2.5 - 0.5x & \text{if } 13 > x \end{cases}$




**69.** $[f^{-1} \circ g^{-1}](x) = \frac{x + 2}{16}$

**70.** $[g^{-1} \circ f^{-1}](x) = \frac{x - 44}{16}$

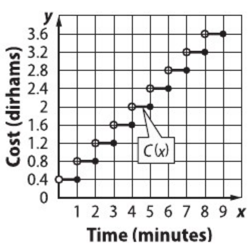
**71.** $(f \circ g)^{-1}(x) = \frac{x - 44}{16}$

**72.** $(g \circ f)^{-1}(x) = \frac{x + 2}{16}$

**73.** $(f \cdot g)^{-1}(x) = \frac{\sqrt{x + 49} - 5}{4}, x \geq -1.25$

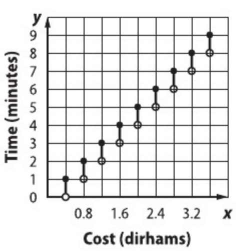
**74.** $[f^{-1} \cdot g^{-1}](x) = \frac{x^2 - 2x - 24}{16}$

**81a.**




**81b.** D = $\{x \mid x \in \mathbb{R}\}$,
R = $\{y \mid y$ positive multiples of 0.4$\}$

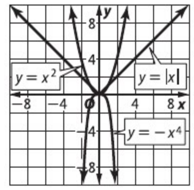
**81c.**




**81d.** D = $\{x \mid x$ positive multiples of 0.4$\}$
R = $\{y \mid y \in \mathbb{R}\}$

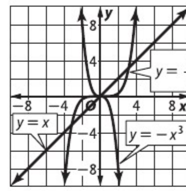
**81e.** The inverse gives the number of possible minutes spent using the scanner that costs $x$ dirhams.

**82a.** No; sample answer:



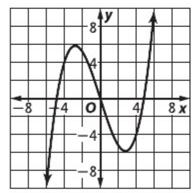

**82b.** True; sample answer: The pattern indicates that no even functions have inverses. When a function is even, $f(x) = f(-x)$. Two $x$-values share a common $y$-value. This violates the horizontal line test. Therefore, the statement is true, and no even functions have inverse functions.

**82c.** Yes; sample answer:




**82d.** Sample answer: The pattern indicates that all odd functions have inverses. While the pattern of the three graphs presented indicates that *some* odd functions have inverses, this is in fact a false statement. If a function is odd, then $f(-x) = -f(x)$ for all $x$. An example of an odd function is $f(x) = \frac{2}{15}x^3 - \frac{47}{15}x$.

Notice from the graph of this function that, while it is odd, it fails the horizontal line test and, therefore, does not have an inverse function.




**84.** Sample answer: The domain of a quadratic function needs to be restricted so that only half of the parabola is shown. The cut-off point of the restriction will be along the axis of symmetry of the parabola. This essentially cuts the parabola into two equal halves. The restriction will be $\left(\frac{-b}{2a}, \infty\right)$ or $\left(-\infty, \frac{-b}{2a}\right)$ for $f(x) = ax^2 + bx + c$.

**85.** False; sample answer: Constant functions are linear, but they do not pass the horizontal line test. Therefore, constant functions are not one-to-one functions and do not have inverse functions.

**87.** Yes; sample answer: One function that does this is $f(x) = \frac{1}{x}$. Even though both limits approach 0, they do it from opposite sides of 0 and no $x$-values ever share a corresponding $y$-value. Therefore, the function passes the horizontal line test.

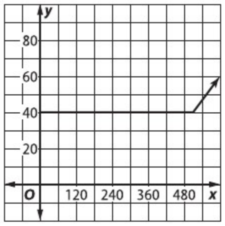
**88.** Sample answer: If the $\pm$ sign is used, then $f(x)$ will no longer be a function because it violates the vertical line test.

**89.** Sample answer: If $f(x) = x^2$, $f$ does not have an inverse because it is not one-to-one. If the domain is restricted to $x \geq 0$, then the function is now one-to-one and $f^{-1}$ exists; $f^{-1}(x) = \sqrt{x}$.

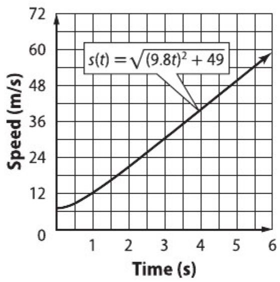
## Study Guide and Review

**61a.** $f(x) = \begin{cases} 39.99 & \text{if } 0 \leq x \leq 500 \\ 39.99 + 0.2(x - 500) & \text{if } x > 500 \end{cases}$

**61c.**




**65.**




Chapter 11 Answer Appendix