مؤسسة الإمارات للتعليم المدرسي
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UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

G12 Adv Term 1
End of Term 2023-24

Justin Dsouza

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المدرسة الإماراتية
مدرسة العروبة للتعليم الثانوي
Justin Dsouza
Teacher

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Justin Daryl Dsouza

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المدرسة الإماراتية
مدرسة العروبة للتعليم الثانوي
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Number of MCQ عدد الأسئلة الموضوعية	15
Marks of MCQ درجة الأسئلة الموضوعية	4
Number of FRQ عدد الأسئلة المقالية	5
Marks per FRQ الدرجات للأسئلة المقالية	(7-11)
Type of All Questions نوع كافة الأسئلة	MCQ/ الأسئلة الموضوعية FRQ/ الأسئلة المقالية
Maximum Overall Grade الدرجة القصوى الممكنة	100
Exam Duration - مدة الامتحان	150 minutes
Mode of Implementation -طريقة التطبيق	SwiftAssess & Paper-Based
Calculator	Allowed
الآلة الحاسبة	مسموحة

Question* السؤال*	Learning Outcome/Performance Criteria** ناتج التعلم/ معاييرالأداء**	Reference(s) in the Student Book (English Version) المرجع في كتاب الطالب (النسخة الانجليزية) Example/Exercise مثال/تمرين	Page الصفحة
1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68
2	Find a limit algebraically or graphically, if it exists. إيجاد قيمة نهاية دالة ما جبريا وبيانيا، إن وجدت	(7-10)	75
3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85
4	Determine the continuity of a function at a given point. البحث في اتصال دالة عند نقطة معطاة	(15-28)	95
5	Find horizontal, vertical, and slant asymptotes using limits. إيجاد خطوط التقارب الأفقية والرأسية والمائلة باستخدام النهايات	(23-32) (51-56)	106
6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically. فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	(23-29) 30	141 142
7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

Question* السؤال*	Learning Outcome/Performance Criteria** ناتج التعلم/ معاييرالأداء**	Reference(s) in the Student Book (English Version) المرجع في كتاب الطالب (النسخة الانجليزية)	
		Example/Exercise مثال/تمرين	Page الصفحة
8	Understand the relationship between continuity and differentiability.	(19-22)	151
	فهم العلاقة بين الاتصال والاشتقاق	32	152
9	Find the derivative of a function at a given point using the Power Rule.	(33-38)	161
	إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة		
10	Use differentiation rules and higher derivatives in solving real-life problems.	(21-26)	161
	استخدام قواعد الاشتقاق والمشتقات العليا في حل مسائل حياتية		
11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	
12	Find the derivative of an inverse function using the Chain Rule.	(17-22)	176
	إيجاد مشتقة معكوس دالة باستخدام قاعدة السلسلة		
13	Find the derivatives of trigonometric functions using differentiation rules.	(1-22)	184
	إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق		
14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194
15	Use implicit differentiation to find derivatives of inverse trigonometric functions.	(29-34)	204
	استخدام الاشتقاق الضمني في إيجاد مشتقات الدوال المثلثية العكسية		

Question* السؤال*		Learning Outcome/Performance Criteria** ناتج التعلم/ معايير الأداء**	Reference(s) in the Student Book (English Version) المرجع في كتاب الطالب (النسخة الانجليزية) Example/Exercise مثال/تمرين	Page الصفحة
FRQ - الأسئلة المقالية	16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
		a) استخدام نظرية الشطيرة لإيجاد النهايات	37	128
		b) Find limits at infinity and limits that are infinite.	(9-22)	106
		b) إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية	(39-50)	
	17	a) Find the derivative of a function at a given point.	Example2.2	145
		a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
		b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
		b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		
	18	Solve real-life problems using derivatives of exponential and logarithmic functions.	Example 7.5	192
		حل مسائل حياتية باستخدام مشتقات الدوال الأسية واللوغاريتمية الطبيعية	(37,38)	194
	19	Find derivatives implicitly.	Example 8.2	198
		إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
			(13,14)	222
	20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
		التعرف على نظرية القيمة المتوسطة واستخدامها في التطبيقات	(43-46)	220
			(83,84)	223
*	Questions might appear in a different order in the actual exam, or on the exam paper .			
*	قد تظهر الأسئلة بترتيب مختلف في الامتحان الفعلي، أو على ورقة الامتحان .			

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Part 1: MCQ

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Question 1

Estimate an arc length of a given function

Page 68
Exercise 7 - 12

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1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

7. $f(x) = \cos x, 0 \le x \le \pi/2$

(a) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 1)	(0.393, 0.92)	0.400
(0.393, 0.92)	(0.785, 0.71)	0.449
(0.785, 0.71)	(1.18, 0.383)	0.509
(1.18, 0.383)	(1.571, 0)	0.548
	Total	1.906

(b) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{16}$, $\frac{\pi}{8}$, $\frac{3\pi}{16}$, $\frac{\pi}{4}$, $\frac{5\pi}{16}$, $\frac{3\pi}{8}$, $\frac{7\pi}{16}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 1)	(0.196, 0.98)	0.197
(0.196, 0.98)	(0.393, 0.92)	0.204
(0.393, 0.92)	(0.589, 0.83)	0.217
(0.589, 0.83)	(0.785, 0.71)	0.232
(0.785, 0.71)	(0.982, 0.56)	0.248
(0.982, 0.56)	(1.178, 0.38)	0.262
(1.178, 0.38)	(1.37, 0.195)	0.272
(1.37, 0.195)	(1.571, 0)	0.277
	Total	1.909

(c) Actual length approximately 1.9101.

1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

8. $f(x) = \sin x, 0 \le x \le \pi/2$

(a) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 0)	(0.393, 0.38)	0.548
(0.393, 0.38)	(0.785, 0.71)	0.509
(0.785, 0.71)	(1.18, 0.924)	0.449
(1.18, 0.924)	(1.57, 1)	0.400
	Total	1.906

(c) Actual length approximately 1.9101.

(b) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{16}$, $\frac{\pi}{8}$, $\frac{3\pi}{16}$, $\frac{\pi}{4}$, $\frac{5\pi}{16}$, $\frac{3\pi}{8}$, $\frac{7\pi}{16}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 0)	(0.196, 0.2)	0.277
(0.196, 0.2)	(0.39, 0.38)	0.272
(0.39, 0.38)	(0.589, 0.56)	0.262
(0.589, 0.56)	(0.785, 0.71)	0.248
(0.785, 0.71)	(0.982, 0.83)	0.232
(0.982, 0.83)	(1.18, 0.924)	0.217
(1.18, 0.924)	(1.374, 0.98)	0.204
(1.374, 0.98)	(1.571, 1)	0.197
	Total	1.909

1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

9. $f(x) = \sqrt{x+1}, 0 \le x \le 3$

(a)

Left	Right	Length
(0, 1)	(0.75, 1.323)	0.817
(0.75, 1.323)	(1.5, 1.581)	0.793
(1.5, 1.581)	(2.25, 1.803)	0.782
(2.25, 1.803)	(3, 2)	0.776
	Total	3.167

(b)

Left	Right	Length
(0, 1)	(0.375, 1.17)	0.413
(0.375, 1.17)	(0.75, 1.323)	0.404
(0.75, 1.323)	(1.125, 1.46)	0.399
(1.125, 1.46)	(1.5, 1.58)	0.395
(1.5, 1.58)	(1.88, 1.696)	0.392
(1.88, 1.696)	(2.25, 1.80)	0.390
(2.25, 1.80)	(2.63, 1.904)	0.388
(2.63, 1.904)	(3, 2)	0.387
	Total	3.168

(c) Actual length approximately 3.168.

1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

10. $f(x) = 1/x, 1 \le x \le 2$

(a)

Left	Right	Length
(1, 1)	(1.25, 0.8)	0.3202
(1.25, 0.8)	(1.5, 0.67)	0.2833
(1.5, 0.67)	(1.75, 0.571)	0.2675
(1.75, 0.571)	(2, 0.5)	0.2600
	Total	1.1310

(b)

Left	Right	Length
(1, 1)	(1.125, 0.89)	0.167
(1.125, 0.89)	(1.25, 0.8)	0.153
(1.25, 0.8)	(1.375, 0.73)	0.145
(1.375, 0.73)	(1.5, 0.67)	0.139
(1.5, 0.67)	(1.625, 0.62)	0.135
(1.625, 0.62)	(1.75, 0.57)	0.133
(1.75, 0.57)	(1.875, 0.53)	0.131
(1.875, 0.53)	(2, 0.5)	0.129
	Total	1.132

(c) Actual length approximately 1.1321.

1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

11. $f(x) = x^2 + 1, -2 \le x \le 2$

(a)

Left	Right	Length
(-2, 5)	(-1, 2)	3.162
(-1, 2)	(0, 1)	1.414
(0, 1)	(1, 2)	1.414
(1, 2)	(2, 5)	3.162
	Total	9.153

(b)

Left	Right	Length
(-2, 5)	(-1.5, 3.25)	1.820
(-1.5, 3.25)	(-1, 2)	1.346
(-1, 2)	(-0.5, 1.25)	0.901
(-0.5, 1.25)	(0, 1)	0.559
(0, 1)	(0.5, 1.25)	0.559
(0.5, 1.25)	(1, 2)	0.901
(1, 2)	(1.5, 3.25)	1.346
(1.5, 3.25)	(2, 5)	1.820
	Total	9.253

(c) Actual length approximately 9.2936.

1	Estimate an arc length of a given function. تقدير طول القوس على منحنى دالة معطاة	(7-12)	68

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

12. $f(x) = x^3 + 2, -1 \le x \le 1$

(a)

Left	Right	Length
(-1, 1)	(-0.5, 1.875)	1.0078
(-0.5, 1.875)	(0, 2)	0.5154
(0, 2)	(0.5, 2.125)	0.5154
(0.5, 2.125)	(1, 3)	1.0078
	Total	3.0463

(b)

Left	Right	Length
(-1, 1)	(-0.75, 1.58)	0.630
(-0.75, 1.58)	(-.5, 1.88)	0.388
(-.5, 1.88)	(-0.25, 1.98)	0.273
(-0.25, 1.98)	(0, 2)	0.251
(0, 2)	(0.25, 2.016)	0.251
(0.25, 2.016)	(0.5, 2.13)	0.273
(0.5, 2.13)	(0.75, 2.42)	0.388
(0.75, 2.42)	(1, 3)	0.630
	Total	3.084

(c) Actual length approximately 3.0957.

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Question 2

Find a limit algebraically or graphically, if it exists

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Exercise 7 - 10

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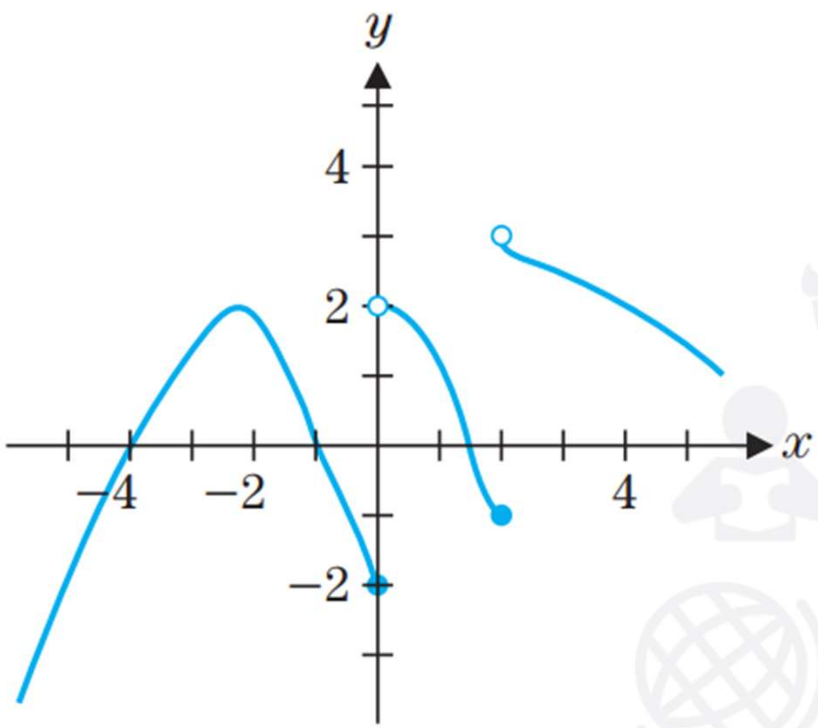
2	Find a limit algebraically or graphically, if it exists.	(7-10)	75
	إيجاد قيمة نهاية دالة ما جبريا وبيانيا، إن وجدت		

In exercises 7 and 8, identify each limit or state that it does not exist.



7. (a) $\lim_{x\to0^-} f(x)$ (b) $\lim_{x\to0^+} f(x)$ (c) $\lim_{x\to0} f(x)$

(d) $\lim_{x\to-2^-} f(x)$ (e) $\lim_{x\to-2^+} f(x)$ (f) $\lim_{x\to-2} f(x)$

(g) $\lim_{x\to-1} f(x)$ (h) $\lim_{x\to1^-} f(x)$

(a) $\lim_{x\to0^-} f(x) = -2$

(b) $\lim_{x\to0^+} f(x) = 2$

(c) $\lim_{x\to0} f(x)$ Does not exist.

(d) $\lim_{x\to-2^-} f(x) = 2$

(e) $\lim_{x\to-2^+} f(x) = 2$

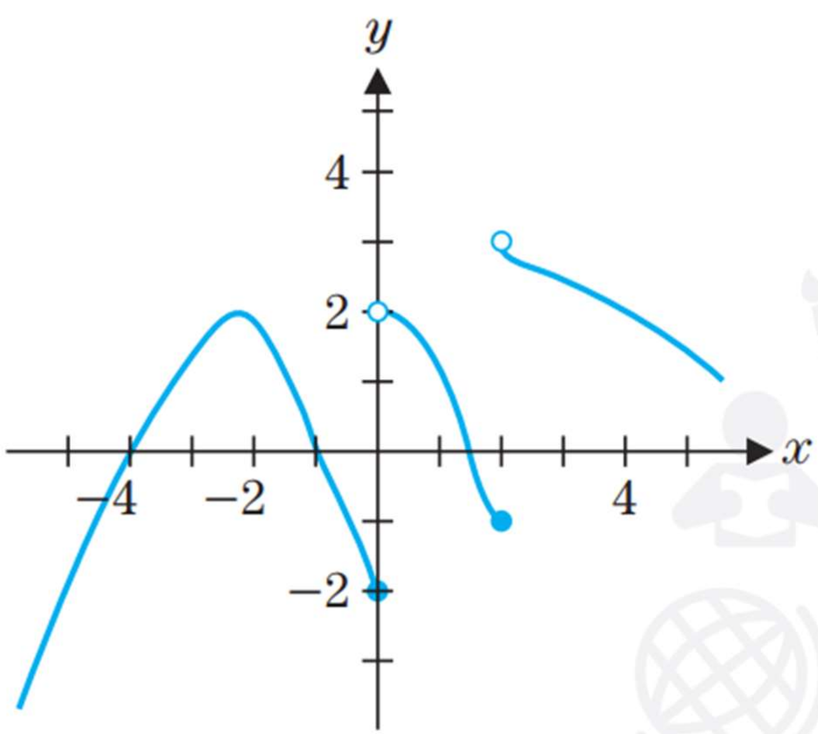
(f) $\lim_{x\to-2} f(x) = 2$

(g) $\lim_{x\to-1} f(x) = 0$

(h) $\lim_{x\to-3} f(x) = 1$

2	Find a limit algebraically or graphically, if it exists. إيجاد قيمة نهاية دالة ما جبريا وبيانيا، إن وجدت	(7-10)	75

In exercises 7 and 8, identify each limit or state that it does not exist.



8. (a) $\lim\limits_{x\to 1^-} f(x)$ (b) $\lim\limits_{x\to 1^+} f(x)$ (c) $\lim\limits_{x\to 1} f(x)$

(d) $\lim\limits_{x\to 2^-} f(x)$ (e) $\lim\limits_{x\to -2^+} f(x)$ (f) $\lim\limits_{x\to 2} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$ (h) $\lim\limits_{x\to -3} f(x)$

(a) $\lim\limits_{x\to 1^-} f(x) = 1$

(b) $\lim\limits_{x\to 1^+} f(x) = 1$

(c) $\lim\limits_{x\to 1} f(x) = 1$

(d) $\lim\limits_{x\to 2^-} f(x) = -1$

(e) $\lim\limits_{x\to 2^+} f(x) = 3$

(f) $\lim\limits_{x\to 2} f(x)$ does not exist.

(g) $\lim\limits_{x\to 3^-} f(x) = 2.5$

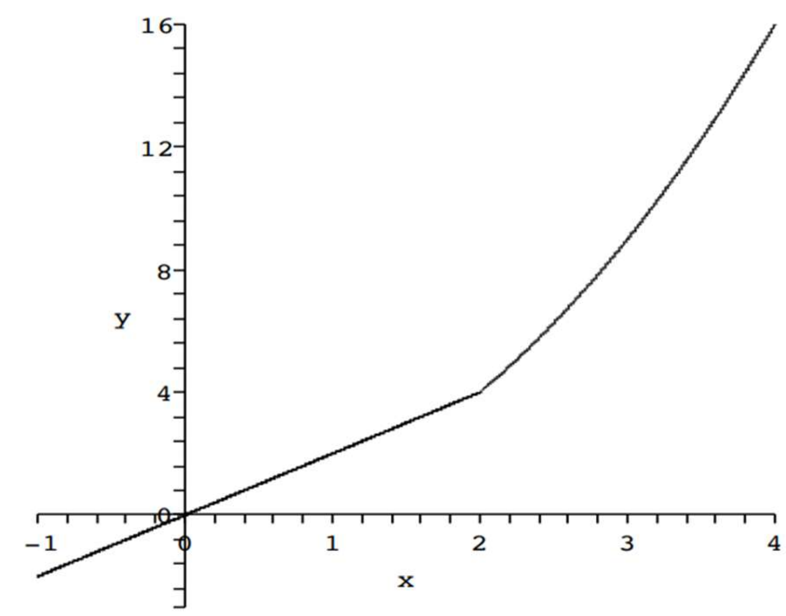
(h) $\lim\limits_{x\to -3} f(x) = 1.5$

2	Find a limit algebraically or graphically, if it exists. إيجاد قيمة نهاية دالة ما جبريا وبيانيا، إن وجدت	(7-10)	75

9. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ and identify each limit.

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to 2^+} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$

(d) $\lim\limits_{x\to 1} f(x)$ (e) $\lim\limits_{x\to 3} f(x)$



(a) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 2x = 4$

(b) $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} x^2 = 4$

(c) $\lim\limits_{x\to 2} f(x) = 4$

(d) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} 2x = 2$

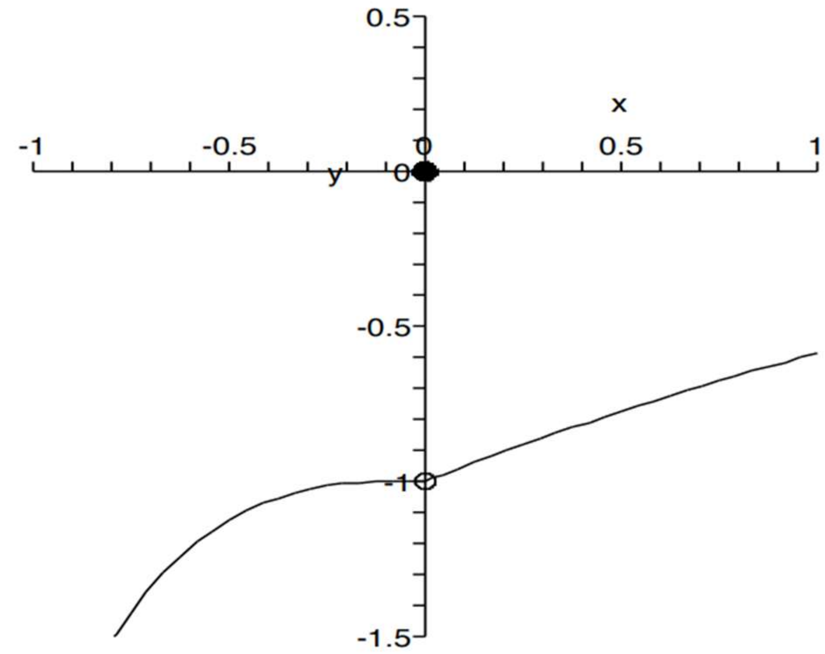
(e) $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} x^2 = 3^2 = 9$

2	Find a limit algebraically or graphically, if it exists. إيجاد قيمة نهاية دالة ما جبريا وبيانيا، إن وجدت	(7-10)	75

10. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ \sqrt{x+1} - 2 & \text{if} \quad x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to -1} f(x)$ (e) $\lim_{x \to 1^-} f(x)$



(a) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x^3 - 1 = -1$

(b) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sqrt{x+1} - 2$
$= -1$

(c) $\lim\limits_{x \to 0} f(x) = -1$

(d) $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} x^3 - 1 = -2$

(e) $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \sqrt{x+1} - 2 = 0$

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Question 3

Find limits of polynomial, rational, and trigonometric functions using theorems

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Exercise 1 - 28

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Justin Dsouza
Teacher

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists.

Assume that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$

1. $\lim\limits_{x \to 0}(x^2 - 3x + 1)$

$$\lim_{x \to 0}(x^2 - 3x + 1) = 0^2 - 3(0) + 1 = 1$$

2. $\lim\limits_{x \to 2} \sqrt[3]{2x+1}$

$$\lim_{x \to 2} \sqrt[3]{2x+1} = \sqrt[3]{2(2)+1} = \sqrt[3]{5}.$$

3. $\lim\limits_{x \to 0} \cos^{-1}(x^2)$

$$\lim_{x \to 0} \cos^{-1}(x^2) = \cos^{-1} 0 = \frac{\pi}{2}.$$

4. $\lim\limits_{x \to 2} \dfrac{x-5}{x^2+4}$

$$\lim_{x \to 2} \frac{x-5}{x^2+4} = \frac{2-5}{2^2+4} = -\frac{3}{8}$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists.

Assume that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1.$

5. $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = 3 + 2 = 5$$

6. $\lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 3x + 2}$

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{(x+2)}{(x-2)} = \frac{3}{-1} = -3.$$

7. $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{x+1}{x+2} = \frac{2+1}{2+2} = \frac{3}{4}$$

8. $\lim\limits_{x\to 1} \dfrac{x^3 - 1}{x^2 + 2x - 3}$

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 + 2x - 3} = \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x^2 + x + 1}{x + 3} = \frac{1^2 + 1 + 1}{1 + 3} = \frac{3}{4}$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

9. $\lim\limits_{x\to 0} \frac{\sin x}{\tan x}$

$$\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\sin x}{\frac{\sin x}{\cos x}}$$
$$= \lim_{x\to 0} \cos x = \cos 0 = 1$$

10. $\lim\limits_{x\to 0} \frac{\tan x}{x}$

$$\lim_{x\to 0} \frac{\tan x}{x}$$
$$= \lim_{x\to 0} \frac{\sin x}{x\cos x}$$
$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) = 1.$$

11. $\lim\limits_{x\to 0} \frac{xe^{-2x+1}}{x^2+x}$

$$\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2+x}$$
$$= \lim_{x\to 0} \frac{x(e^{-2x+1})}{x(x+1)}$$
$$= \lim_{x\to 0} \frac{e^{-2x+1}}{x+1} = \frac{e^{-2(0)+1}}{0+1} = e$$

12. $\lim\limits_{x\to 0} x^2 \csc^2 x$

$$\lim_{x\to 0} x^2 \csc^2 x = \lim_{x\to 0} \frac{x^2}{\sin^2 x}$$
$$= \left(\lim_{x\to 0} \frac{1}{\frac{\sin x}{x}}\right)\left(\lim_{x\to 0} \frac{1}{\frac{\sin x}{x}}\right) = 1$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

13. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$$
$$=\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$$
$$=\lim_{x\to 0}\frac{x+4-4}{x(\sqrt{x+4}+2)}$$
$$=\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$$
$$=\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\frac{1}{4}$$

14. $\lim_{x\to 0}\frac{2x}{3-\sqrt{x+9}}$

$$\lim_{x\to 0}\frac{2x}{3-\sqrt{x+9}}$$
$$=\lim_{x\to 0}\frac{2x}{(3-\sqrt{x+9})}\frac{(3+\sqrt{x+9})}{(3+\sqrt{x+9})}$$
$$=\lim_{x\to 0}\frac{2x(3+\sqrt{x+9})}{-x}$$
$$=\lim_{x\to 0}-2(3+\sqrt{x+9})=-12$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

15. $\lim\limits_{x\to 1} \frac{x-1}{\sqrt{x}-1}$

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$
$$= \lim_{x\to 1} \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}$$
$$= \lim_{x\to 1}(\sqrt{x}+1) = \sqrt{1}+1 = 2$$

16. $\lim\limits_{x\to 4} \frac{x^3-64}{x-4}$

$$\lim_{x\to 4} \frac{x^3-64}{x-4} = \lim_{x\to 4} \frac{(x-4)\left(x^2+4x+16\right)}{(x-4)}$$
$$= \lim_{x\to 4} \left(x^2+4x+16\right)$$
$$= 4^2+4\times 4+16 = 48$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists.

Assume that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1.$

17. $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$$
$$=\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}\right)$$
$$=\lim_{x\to 1}\left(\frac{x+1}{(x-1)(x+1)}-\frac{2}{(x-1)(x+1)}\right)$$
$$=\lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)$$
$$=\lim_{x\to 1}\left(\frac{1}{x+1}\right)=\frac{1}{2}$$

18. $\lim\limits_{x\to 0}\left(\dfrac{2}{x}-\dfrac{2}{|x|}\right)$

Undefined. The limit from the right is 0, but the limit from the left does not exist.

19. $\lim\limits_{x\to 0}\dfrac{1-e^{2x}}{1-e^{x}}$

$$\lim_{x\to 0}\frac{1-e^{2x}}{1-e^{x}}$$
$$=\lim_{x\to 0}\frac{\left(1-e^{x}\right)\left(1+e^{x}\right)}{1-e^{x}}$$
$$=\lim_{x\to 0}\left(1+e^{x}\right)=2$$

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

20. $\lim\limits_{x\to 0} \frac{\sin |x|}{x}$

$$\lim_{x\to 0^+} \frac{\sin(|x|)}{x} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0^-} \frac{\sin(|x|)}{x}$$

$$= \lim_{x\to 0^-} \frac{\sin(-x)}{x}$$

$$= \lim_{x\to 0^-} \frac{-\sin(x)}{x} = -1$$

Since the limit from the left does not equal the limit from the right, we see that $\lim\limits_{x\to 0} \frac{\sin(|x|)}{x}$ does not exist.

21. $\lim\limits_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x = 2(2) = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 2^2 = 4$$

$$\lim_{x\to 2} f(x) = 4$$

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

22. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 3x+1 & \text{if } x \geq -1 \end{cases}$

Undefined. The limit from the left is 2, but the limit from the right is -2.

23. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2x+1) = 2(-1)+1 = -1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 3 = 3$$

Therefore $\lim_{x\to -1} f(x)$ does not exist.

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

24. $\lim\limits_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = 3,$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x+1 = 3,$$
$$\text{Therefore } \lim_{x\to 1} f(x) = 3.$$

25. $\lim\limits_{h\to 0} \dfrac{(2+h)^2 - 4}{h}$

$$\lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$$
$$= \lim_{h\to 0} \frac{(4+4h+h^2) - 4}{h}$$
$$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} 4+h = 4$$

26. $\lim\limits_{h\to 0} \dfrac{(1+h)^3 - 1}{h}$

$$\lim_{h\to 0} \frac{(1+h)^3 - 1}{h}$$
$$= \lim_{h\to 0} \frac{1+3h+3h^2+h^3-1}{h}$$
$$= \lim_{h\to 0} \frac{h(3+3h+h^2)}{h}$$
$$= \lim_{h\to 0} 3+3h+h^2 = 3$$

3	Find limits of polynomial, rational, and trigonometric functions using theorems. إيجاد نهاية الدوال كثيرة الحدود والنسبية والمثلثية باستخدام نظريات النهايات	(1-28)	85

In exercises 1–28, evaluate the indicated limit, if it exists.

Assume that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

27. $\lim\limits_{x \to 2} \dfrac{\sin(x^2 - 4)}{x^2 - 4}$

Consider $f(x) = \dfrac{\sin x}{x}$ and a polynomial

$p(x) = x^2 - 4$ such that $p(2) = 0$.

Also $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

Therefore by the theorem 3.4(viii),

$\lim\limits_{x \to a} f(p(x)) = L$

$\Rightarrow \lim\limits_{x \to 2} \dfrac{\sin(x^2 - 4)}{x^2 - 4} = 1$.

28. $\lim\limits_{x \to 0} \dfrac{\tan x}{5x}$

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan x}{5x} &= \lim_{x \to 0} \frac{\sin x}{5x \cos x} \\
&= \lim_{x \to 0} \left(\frac{1}{5} \frac{\sin x}{x} \frac{1}{\cos x} \right) \\
&= \frac{1}{5} \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \frac{1}{\cos x} \right) \\
&= \frac{1}{5}(1)(1) = \frac{1}{5}
\end{aligned}$$

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Question 4

Determine the continuity of a function at a given point

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Exercise 15 - 28

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Teacher

4	Determine the continuity of a function at a given point. البحث في اتصال دالة عند نقطة معطاة	(15-28)	95

In exercises 15–20, explain why each function fails to be continuous at the given x-value by indicating which of the three conditions in Definition 4.1 are not met.

15. $f(x) = \dfrac{x}{x-1}$ at $x = 1$

$f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

16. $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$

Discontinuous because function is not defined at $x = 1$.

17. $f(x) = \sin\dfrac{1}{x}$ at $x = 0$

$f(0)$ is not defined and $\lim_{x\to 0} f(x)$ does not exist.

18. $f(x) = \dfrac{e^{x-1}}{e^x - 1}$ at $x = 0$

The function is discontinuous at $x = 0$, as it is not defined at $x = 0$.

4	Determine the continuity of a function at a given point. البحث في اتصال دالة عند نقطة معطاة	(15-28)	95

In exercises 15–20, explain why each function fails to be continuous at the given x-value by indicating which of the three conditions in Definition 4.1 are not met.

19. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2) = 4$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2) = 4$$
$$\lim_{x \to 2} f(x) = 4; f(2) = 3$$
$$\lim_{x \to 2} f(x) \neq f(2)$$

20. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

Discontinuous because function is not defined at $x = 2$.

4	Determine the continuity of a function at a given point. البحث في اتصال دالة عند نقطة معطاة	(15-28)	95

In exercises 21–28, determine the intervals on which f is continuous.

21. $f(x) = \sqrt{x+3}$

Continuous where $x+3>0$, i.e. on $(-3, \infty)$

22. $f(x) = \sqrt{x^2-4}$

Continuous where $x^2-4>0$, i.e. on $(\infty, -2)$ and $(2, \infty)$.

23. $f(x) = \sqrt[3]{x+2}$

Continuous everywhere, i.e. on $(-\infty, \infty)$.

24. $f(x) = (x-1)^{3/2}$

Continuous where $x-1>0$, i.e. on $(1, \infty)$.

25. $f(x) = \sin^{-1}(x+2)$

$\sin^{-1}(x+2)$ is continuous on interval [-3, -1].

26. $f(x) = \ln(\sin x)$

$\ln(\sin x)$ is continuous whenever $\sin x > 0$, that is on the interval $(2k\pi, \ (2k+1)\pi)$ for all integral values of k.

4	Determine the continuity of a function at a given point. البحث في اتصال دالة عند نقطة معطاة	(15-28)	95

In exercises 21–28, determine the intervals on which f is continuous.

27. $f(x) = \dfrac{\sqrt{x+1} + e^x}{x^2 - 2}$

$f(x)$ is continuous, on interval $[-1, \infty)$ whenever $x \neq \sqrt{2}$.

28. $f(x) = \dfrac{\ln(x^2 - 1)}{\sqrt{x^2 - 2x}}$

$f(x)$ is continuous on the intervals $(-\infty, -1)$ and $(2, \infty)$

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Question 5

Find horizontal, vertical, and slant asymptotes using limits

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Exercise 23 – 32 & 51 - 56

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Teacher

In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

23. (a) $f(x) = \dfrac{x}{4 - x^2}$

$4 - x^2 = 0 \Rightarrow 4 = x^2$ so we have vertical asymptotes at $x = \pm 2$.

$f(x) \to \infty$ as $x \to -2^-$

$f(x) \to -\infty$ as $x \to -2^+$

$f(x) \to \infty$ as $x \to 2^-$

$f(x) \to -\infty$ as $x \to 2^+$

Again, we have

$$\lim_{x \to \pm\infty} \frac{x}{4 - x^2}$$

$$= \lim_{x \to \pm\infty} \frac{x}{x^2\left(\frac{4}{x^2} - 1\right)}$$

$$= \lim_{x \to \pm\infty} \frac{1}{x\left(\frac{4}{x^2} - 1\right)} = 0.$$

So there is a horizontal asymptote at $y = 0$.

(b) $f(x) = \dfrac{x^2}{4 - x^2}$

Vertical asymptotes at $x = \pm 2$.

$f(x) \to \infty$ as $x \to 2^-$ and $x \to -2^+$.

$f(x) \to -\infty$ as $x \to 2^+$ and $x \to -2^-$.

Horizontal asymptote at $y = -1$.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
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In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

24. (a) $f(x) = \dfrac{x}{\sqrt{4+x^2}}$

Since $4 + x^2$ is never 0, there are no vertical asymptotes. We have

$$\lim_{x\to\infty} \frac{x}{\sqrt{4+x^2}} = \lim_{x\to\infty} \frac{x}{x\sqrt{\frac{4}{x^2}+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{4}{x^2}+1}} = \frac{1}{\sqrt{1}} = 1$$

and

$$\lim_{x\to-\infty} \frac{x}{\sqrt{4+x^2}} = \lim_{x\to-\infty} \frac{x}{-x\sqrt{\frac{4}{x^2}+1}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{\frac{4}{x^2}+1}} = \frac{-1}{\sqrt{1}} = -1,$$

so there are horizontal asymptotes at $y = 1$ and $y = -1$.

(b) $f(x) = \dfrac{x}{\sqrt{4-x^2}}$

The function is only defined in $(-2, 2)$. Two one-sided vertical asymptotes at $x = \pm 2$. $f(x) \to \infty$ as $x \to 2^-$, and $f(x) \to -\infty$ as $x \to -2^+$.
No horizontal asymptotes.

In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

25. $f(x) = \dfrac{3x^2+1}{x^2-2x-3}$

The denominator factors:

$x^2 - 2x - 3 = (x-3)(x+1)$.

Since neither $x = 3$ nor $x = -1$ are zeros of the numerator, we see that $f(x)$ has vertical asymptotes at $x = 3$ and $x = -1$.

$f(x) \to -\infty$ as $x \to 3^-$,
$f(x) \to \infty$ as $x \to 3^+$,
$f(x) \to \infty$ as $x \to -1^-$, and
$f(x) \to -\infty$ as $x \to -1^+$.

We have

$$\lim_{x\to\pm\infty} \frac{3x^2+1}{x^2-2x-3}$$

$$= \lim_{x\to\pm\infty} \frac{3+1/x^2}{1-2/x-3/x^2} = 3.$$

So there is a horizontal asymptote at $y = 3$.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
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In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

26. $f(x) = \dfrac{1-x}{x^2+x-2}$

Vertical asymptote at $x = -2$.

$f(x) \to \infty$ as $x \to -2^-$.
$f(x) \to -\infty$ as $x \to -2^+$ and $x \to -2^-$.
Again, we have

$$\lim_{x\to\pm\infty} \frac{1-x}{x^2+x-2} = 0$$

So there is a horizontal asymptote at $y = 0$.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
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In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

27. $f(x) = 4\tan^{-1} x - 1$

The function is continuous for all x, so no vertical asymptotes. We have

$$\begin{aligned}\lim_{x\to\infty} 4\tan^{-1} x - 1 &= 4(\lim_{x\to\infty} \tan^{-1} x) - 1 \\ &= 4(\pi/2) - 1 \\ &= 2\pi - 1\end{aligned}$$

and

$$\begin{aligned}&\lim_{x\to-\infty} 4\tan^{-1} x - 1 \\ &\quad = 4(\lim_{x\to-\infty} \tan^{-1} x) - 1 \\ &\quad = 4(-\pi/2) - 1 \\ &\quad = -2\pi - 1,\end{aligned}$$

so there are horizontal asymptotes

at $y = 2\pi - 1$ and $y = -2\pi - 1$.

28. $f(x) = \ln(1 - \cos x)$

The function $\ln x$ has a one-sided vertical asymptote at $x = 0$, so $f(x) = \ln(1 - \cos x)$ will have a vertical asymptote whenever

$1 - \cos x = 0$, i.e., whenever $\cos x = 1$.

This happens when $x = 2k\pi$ for any integer k. Since $1 - \cos x \geq 0$ for all x, $f(x)$ is defined at all points except for these vertical asymptotes. Thus as $f(x)$ approaches any of these asymptotes (from either side), it behaves like $\ln x$ approaching 0 from the right, so $f(x) \to -\infty$ as x approaches any of these asymptotes from either side.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
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In exercises 29–32, determine all vertical and slant asymptotes.

29. $y = \dfrac{x^3}{4 - x^2}$

Vertical asymptotes at $x = \pm 2$.

The slant asymptote is $y = -x$.

30. $y = \dfrac{x^2 + 1}{x - 2}$

Vertical asymptote at $x = 2$.

The slant asymptote is $y = x + 2$.

31. $y = \dfrac{x^3}{x^2 + x - 4}$

Vertical asymptotes at $x = \frac{-1 \pm \sqrt{17}}{2}$.

The slant asymptote is $y = x - 1$.

32. $y = \dfrac{x^4}{x^3 + 2}$

Vertical asymptote at $x = -\sqrt[3]{2}$.

The slant asymptote is $y = x$.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
	إيجاد خطوط التقارب الأفقية والرأسية والمائلة باستخدام النهايات	(51-56)	

51. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree of $p(x)$ greater than the degree of $q(x)$. Determine whether $y = f(x)$ has a horizontal asymptote.

Suppose the degree of q is n. If we divide both $p(x)$ and $q(x)$ by x^n, then the new denominator will approach a constant while the new numerator tends to ∞, so there is no horizontal asymptote.

52. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree (largest exponent) of $p(x)$ less than the degree of $q(x)$. Determine the horizontal asymptote of $y = f(x)$.

If the degree of the polynomial in the denominator is larger, the horizontal asymptote is $y = 0$.

5	Find horizontal, vertical, and slant asymptotes using limits.	(23-32)	106
	إيجاد خطوط التقارب الأفقية والرأسية والمائلة باستخدام النهايات	(51-56)	

53. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a slant asymptote $y = x + 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

When we do long division, we get a remainder of $x + 2$, so the degree of p is one greater than the degree of q.

54. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a horizontal asymptote $y = 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

If the horizontal asymptote is $y = 2$, the degrees of the numerator and denominator must be the same.

55. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = -\frac{1}{2}$ and exactly one vertical asymptote $x = 3$.

The function $q(x) = -2(x-2)(x-3)$ satisfies the given conditions.

56. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = 2$ and two vertical asymptotes $x = \pm 3$.

The function $q(x) = \dfrac{x^2}{2} - \dfrac{9}{2}$ satisfies the given conditions.

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Question 6

Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically

Page 141 & 142

Exercise 23 – 29 & 30

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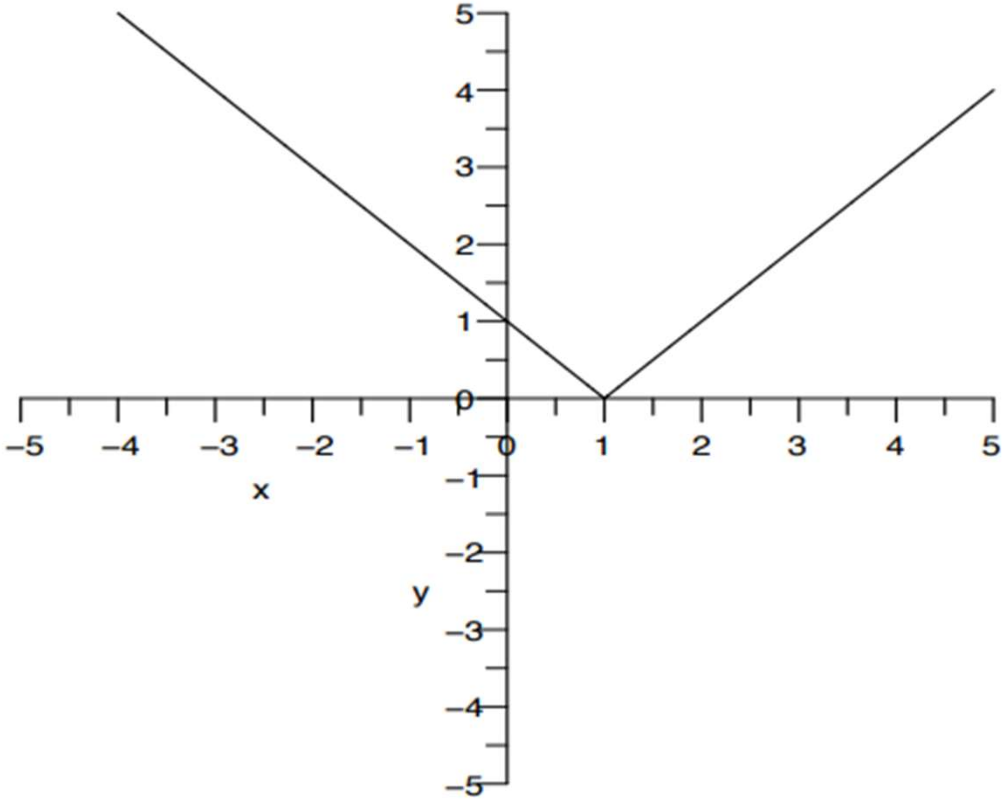
Justin Dsouza
Teacher

6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 23–26, use graphical and numerical evidence to explain why a tangent line to the graph of $y = f(x)$ at $x = a$ does not exist.

23. $f(x) = |x - 1|$ at $a = 1$

A graph makes it apparent that this function has a corner at $x = 1$.



Numerical evidence suggests that,

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = 1$$

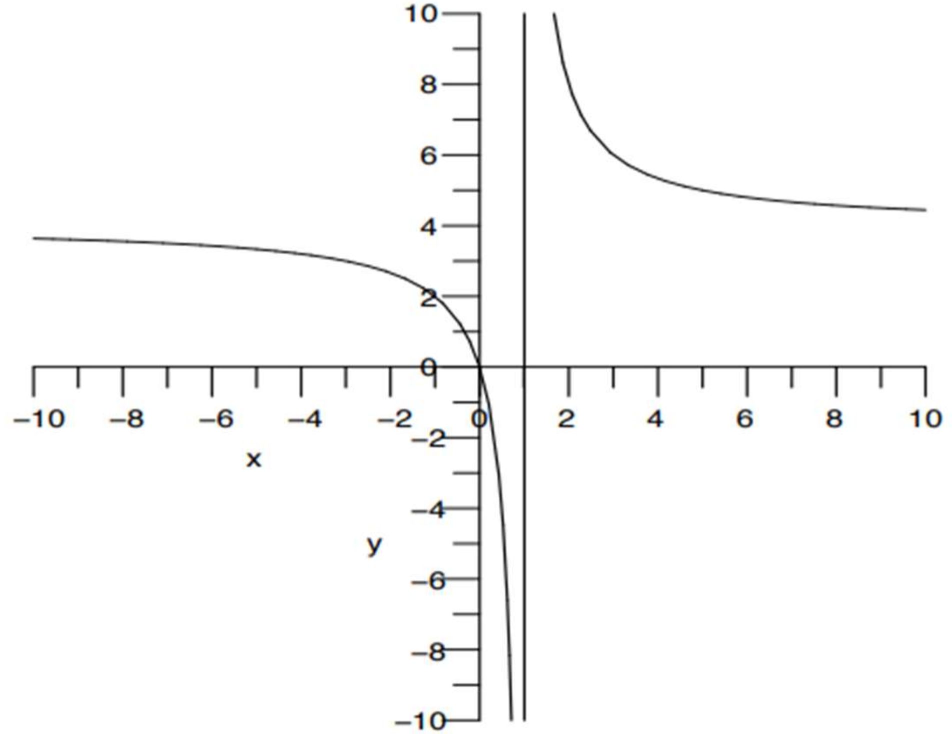
while $\displaystyle\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = -1.$

Since these are not equal, there is no tangent line.

6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيّا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 23–26, use graphical and numerical evidence to explain why a tangent line to the graph of $y = f(x)$ at $x = a$ does not exist.

24. $f(x) = \dfrac{4x}{x-1}$ at $a = 1$



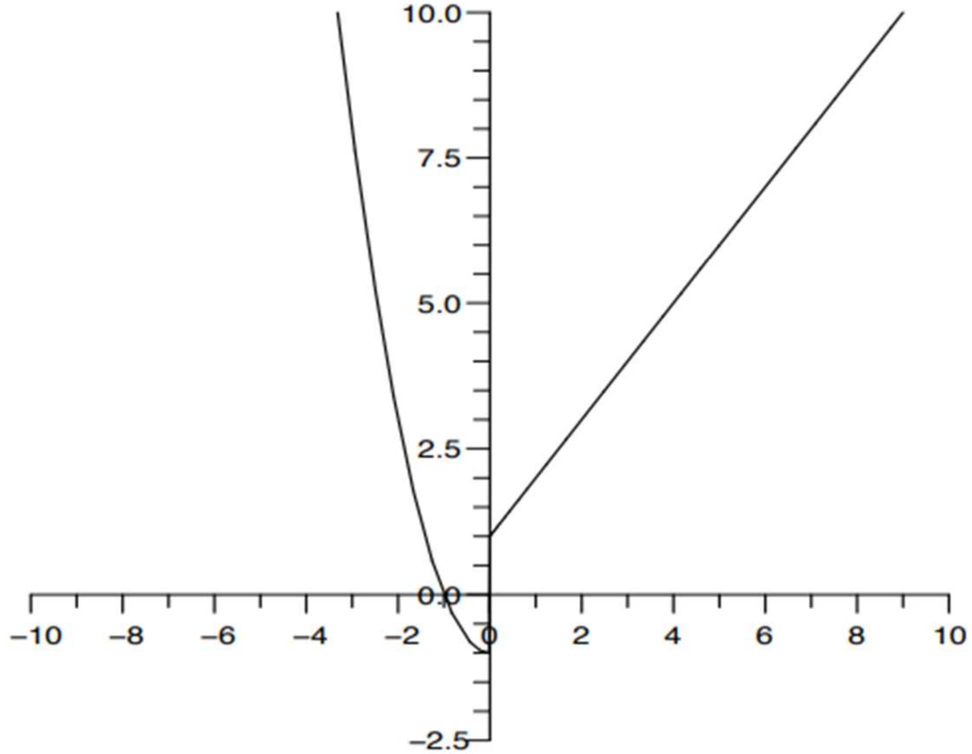
Tangent line does not exist at $x = 1$ because the function is not defined there.

6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 23–26, use graphical and numerical evidence to explain why a tangent line to the graph of $y = f(x)$ at $x = a$ does not exist.

25. $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$ at $a = 0$

From the graph it is clear that, curve is not continuous at $x = 0$ therefore tangent line at $y = f(x)$ at $x = 0$ does not exist.



Also,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{h^2 - 1 - (-1)}{h}$$

$$= \lim_{h\to 0^-} \frac{h^2}{h} = \lim_{h\to 0^-} h = 0$$

Similarly,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h + 1 - (1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Numerical evidence suggest that,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

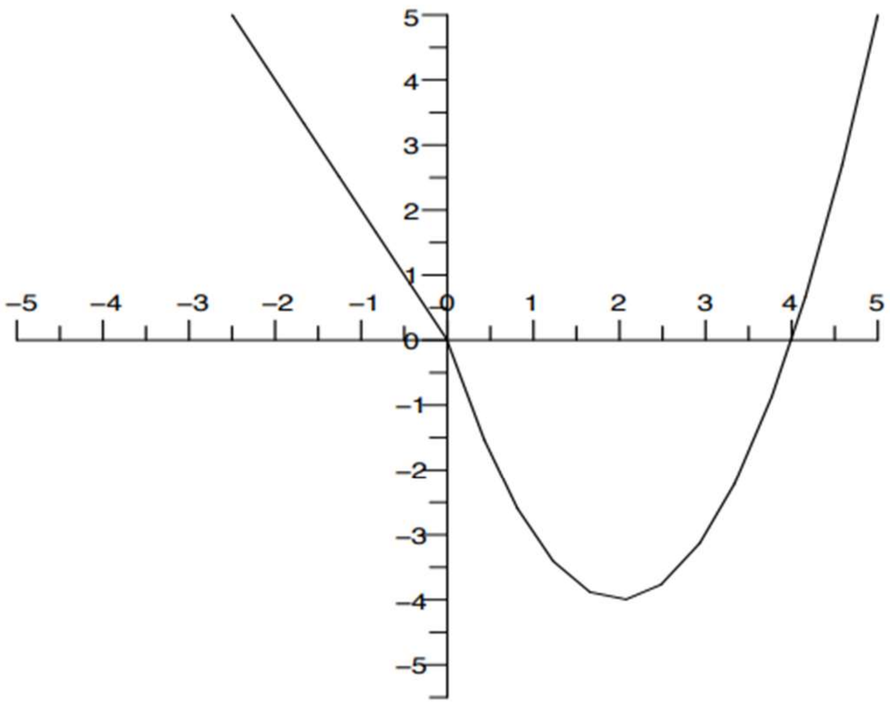
$$\neq \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}.$$

Therefore tangent line does not exist at $x = 0$.

6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 23–26, use graphical and numerical evidence to explain why a tangent line to the graph of $y = f(x)$ at $x = a$ does not exist.

26. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x^2 - 4x & \text{if } x > 0 \end{cases}$ at $a = 0$



From the graph it is clear that, the curve of $y = f(x)$ is not smooth at $x = 0$ therefore tangent line at $x = 0$ does not exist.

Also,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-2h}{h} = -2$$

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} (h-4) = -4.$$

Numerical evidence suggest that,

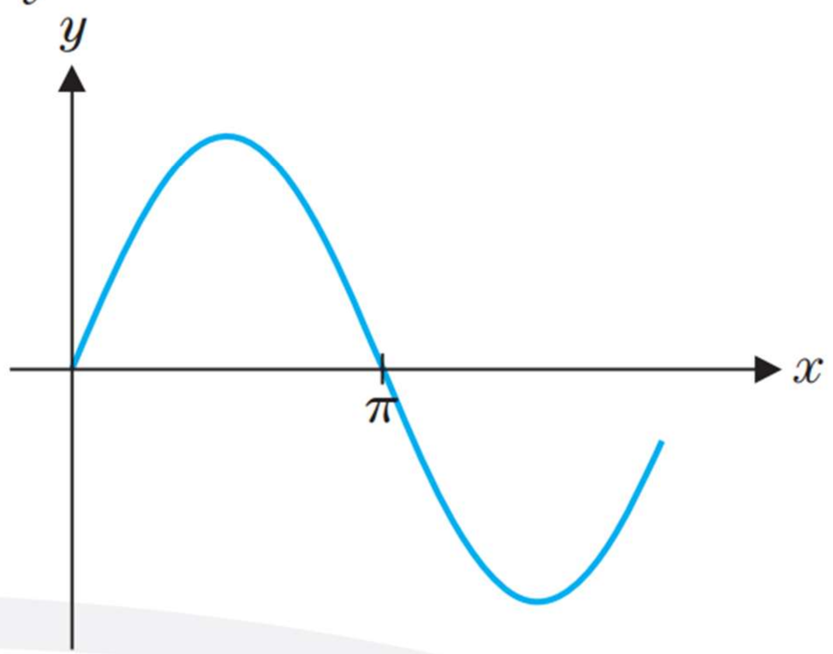
$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}.$$

Therefore tangent line does not exist at $x = 0$.

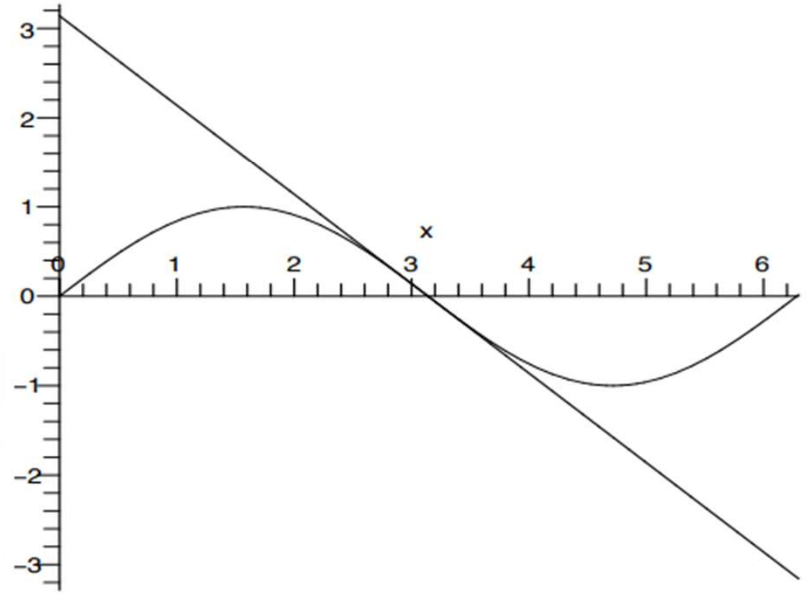
6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 27–30, sketch in a plausible tangent line at the given point, or state that there is no tangent line.

27. $y = \sin x$ at $x = \pi$



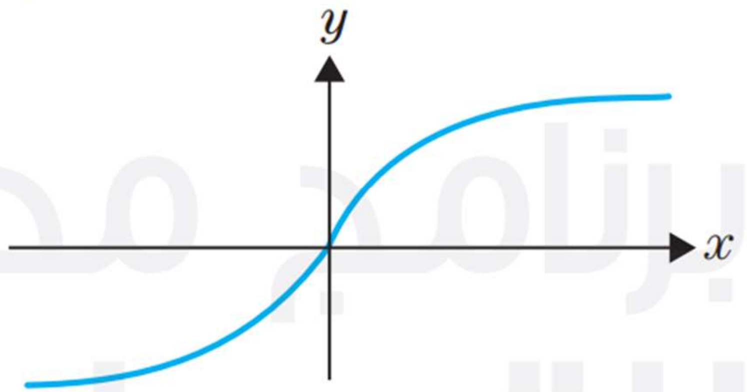
Tangent line at $x = \pi$ to $y = \sin x$ as below:



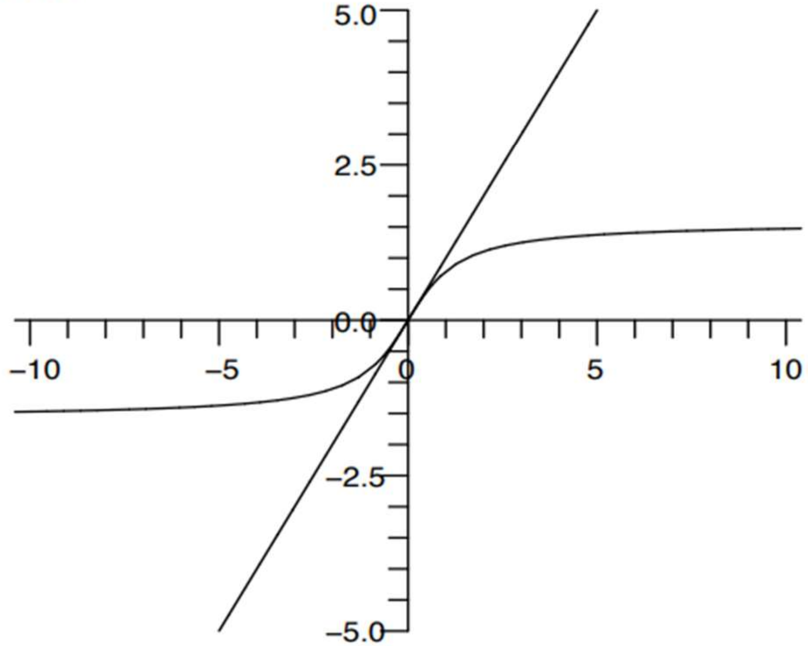
6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 27–30, sketch in a plausible tangent line at the given point, or state that there is no tangent line.

28. $y = \tan^{-1} x$ at $x = 0$



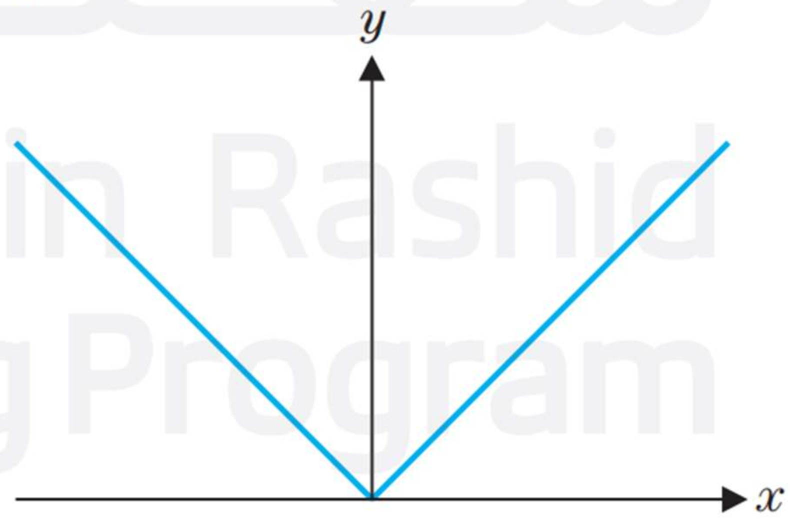
Tangent line at $x = 0$ to $y = \tan^{-1} x$ as below:



6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

In exercises 27–30, sketch in a plausible tangent line at the given point, or state that there is no tangent line.

29. $y = |x|$ at $x = 0$

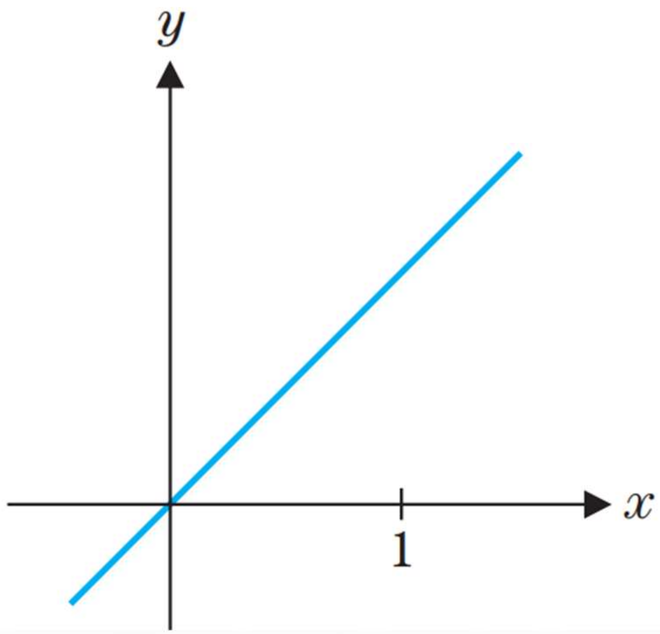


Since the graph has a corner at $x = 0$, tangent line does not exist.

6	Understand the link between the slope of a tangent line and a non-tangent line to a graph geometrically.	(23-29)	141
	فهم العلاقة بين ميل المماس و غير المماس في التمثيل البياني هندسيًا(الربط بين ميل القاطع وميل المماس وتفسيرهما)	30	142

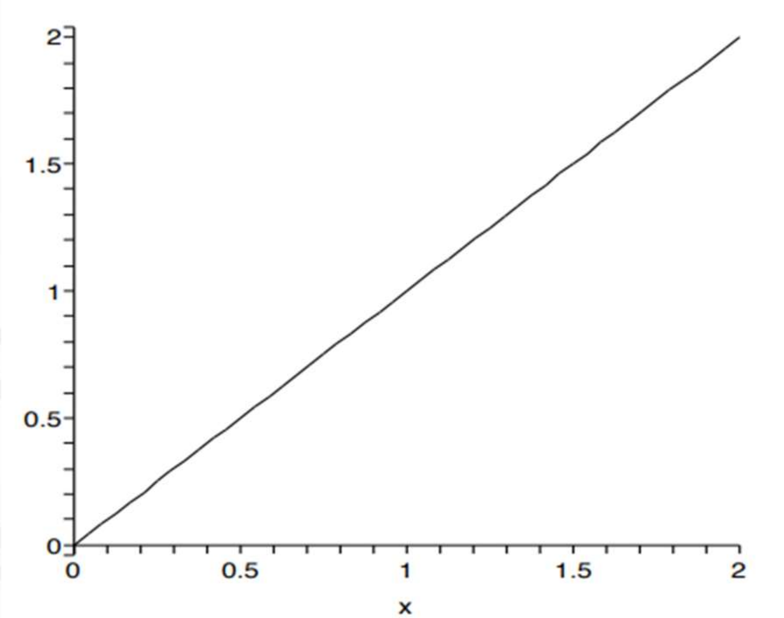
In exercises 27–30, sketch in a plausible tangent line at the given point, or state that there is no tangent line.

30. $y = x$ at $x = 1$



at $x = 1$

The tangent line overlays the line:



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Question 7

Find the average velocity and the instantaneous velocity at a given point

Page 141
Exercise 15 - 22

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Justin Dsouza
Teacher

In exercises 15–18, use the position function *s* (in meters) to find the velocity at time $t = a$ seconds.

15. $s(t) = -4.9t^2 + 5$, (a) $a = 1$; (b) $a = 2$

(a) Velocity at time $t = 1$ is,

$$\lim_{h\to 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h\to 0} \frac{-4.9(1+h)^2 + 5 - (0.1)}{h}$$

$$= \lim_{h\to 0} \frac{-4.9(1+2h+h^2) + 5 - (0.1)}{h}$$

$$= \lim_{h\to 0} \frac{-9.8h - 4.9h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-9.8 - 4.9h)}{h} = -9.8.$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h) - s(2)}{h}$$

$$= \lim_{h\to 0} \frac{-4.9(2+h)^2 + 5 - (-14.6)}{h}$$

$$= \lim_{h\to 0} \frac{-4.9(4+4h+h^2) + 5 - (-14.6)}{h}$$

$$= \lim_{h\to 0} \frac{-19.6h - 4.9h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-19.6 - 4.9h)}{h} = -19.6$$

In exercises 15–18, use the position function s (in meters) to find the velocity at time $t = a$ seconds.

16. $s(t) = 4t - 4.9t^2$, (a) $a = 0$; (b) $a = 1$

(a) Velocity at time $t = 0$ is,

$$\lim_{h\to 0} \frac{s(0+h)-s(0)}{h} = \lim_{h\to 0} \frac{4h-4.9h^2}{h} = \lim_{h\to 0} \frac{h\,(4-4.9h)}{h} = 4 - \lim_{h\to 0} 4.9h = 4.$$

(b) Velocity at time $t = 1$ is,

$$\lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to 0} \frac{4(1+h)-4.9(1+h)^2-(-0.9)}{h}$$
$$= \lim_{h\to 0} \frac{4+4h-4.9-9.8h-4.9h^2+0.9}{h}$$
$$= \lim_{h\to 0} \frac{-5.8h-4.9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h\,(-5.8-4.9h)}{h} = -5.8$$

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 15–18, use the position function s (in meters) to find the velocity at time $t = a$ seconds.

17. $s(t) = \sqrt{t+16}$, (a) $a = 0$; (b) $a = 2$

(a) Velocity at time $t = 0$ is,

$$\lim_{h\to 0} \frac{s(0+h)-s(0)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{h+16}-4}{h} \cdot \frac{\sqrt{h+16}+4}{\sqrt{h+16}+4}$$
$$= \lim_{h\to 0} \frac{(h+16)-16}{h(\sqrt{h+16}+4)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+16}+4} = \frac{1}{8}$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{18+h}-\sqrt{18}}{h}$$

Multiplying by $\dfrac{\sqrt{h+18}+\sqrt{18}}{\sqrt{h+18}+\sqrt{18}}$ gives

$$= \lim_{h\to 0} \frac{(h+18)-18}{h(\sqrt{h+18}+\sqrt{18})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+18}+\sqrt{18}} = \frac{1}{2\sqrt{18}}$$

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 15–18, use the position function s (in meters) to find the velocity at time $t = a$ seconds.

18. $s(t) = 4/t$, (a) $a = 2$; (b) $a = 4$

(a) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h) - s(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{(2+h)} - 2}{h} = \lim_{h\to 0} \frac{\frac{4-4-2h}{(2+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{h(2+h)} = \lim_{h\to 0} \frac{-2}{2+h} = -1.$$

(b) Velocity at time $t = 4$ is,

$$\lim_{h\to 0} \frac{s(4+h) - s(4)}{h} = \lim_{h\to 0} \frac{\frac{4}{(4+h)} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4-1(4+h)}{(4+h)}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{(4+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h(4+h)} = \lim_{h\to 0} \frac{-1}{4+h} = -\frac{1}{4}$$

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 19–22, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

19. $s(t) = 16t^2 + 10$

(a) Points: $(0, 10)$ and $(2, 74)$

Average velocity: $\dfrac{74 - 10}{2} = 32$

(b) Second point: $(1, 26)$

Average velocity: $\dfrac{74 - 26}{1} = 48$

(c) Second point: $(1.9, 67.76)$

Average velocity: $\dfrac{74 - 67.76}{0.1} = 62.4$

(d) Second point: $(1.99, 73.3616)$

Average velocity: $\dfrac{74 - 73.3616}{0.01} = 63.84$

(e) The instantaneous velocity seems to be 64.

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 19–22, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

20. $s(t) = 3t^3 + t$

(a) Points: $(0, 0)$ and $(2, 26)$

Average velocity: $\dfrac{26-0}{2-0} = 13$

(b) Second point: $(1, 4)$

Average velocity: $\dfrac{26-4}{2-1} = 22$

(c) Second point: $(1.9, 22.477)$

Average velocity: $\dfrac{26-22.477}{2-1.9} = 35.23$

(d) Second point: $(1.99, 25.6318)$

Average velocity:

$\dfrac{26-25.6318}{2-1.99} = 36.8203$

(e) The instantaneous velocity seems to be approaching 37.

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 19–22, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

21. $s(t) = \sqrt{t^2 + 8t}$

(a) Points: $(0, 0)$ and $(2, \sqrt{20})$

Average velocity: $\dfrac{\sqrt{20} - 0}{2 - 0} = 2.236068$

(b) Second point: $(1, 3)$

Average velocity: $\dfrac{\sqrt{20} - 3}{2 - 1} = 1.472136$

(c) Second point: $(1.9, \sqrt{18.81})$

Average velocity:

$$\frac{\sqrt{20} - \sqrt{18.81}}{2 - 1.9} = 1.3508627$$

(d) Second point: $(1.99, \sqrt{19.8801})$

Average velocity:

$$\frac{\sqrt{20} - \sqrt{19.88}}{2 - 1.99} = 1.3425375$$

(e) One might conjecture that these numbers are approaching 1.34. The exact limit is $\dfrac{6}{\sqrt{20}} \approx 1.341641$.

7	Find the average velocity and the instantaneous velocity at a given point. إيجاد السرعة المتوسطة والسرعة اللحظية عند نقطة معطاة	(15-22)	141

In exercises 19–22, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

22. $s(t) = 3\sin(t-2)$

(a) Points: $(0, -2.7279)$ and $(2, 0)$
Average velocity:
$$\frac{0-(-2.7279)}{2-0} = 1.3639$$

(b) Second point: $(1, -2.5244)$
Average velocity:
$$\frac{0-(-2.5244)}{2-1} = 2.5244$$

(c) Second point: $(1.9, -0.2995)$
Average velocity:
$$\frac{0-(-0.2995)}{2-1.9} = 2.995$$

(d) Second point: $(1.99, -0.03)$
Average velocity: $\dfrac{0-(-0.03)}{2-1.99} = 3$

(e) The instantaneous velocity seems to be 3.

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Question 8

Understand the relationship between continuity and differentiability

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Exercise 19 – 22 & 32

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Justin Dsouza
Teacher

8	Understand the relationship between continuity and differentiability.	(19-22)	151
	فهم العلاقة بين الاتصال والاشتقاق	32	152

In exercises 19–22, compute the right-hand derivative $D_+f(0) = \lim\limits_{h\to 0^+} \dfrac{f(h)-f(0)}{h}$ and the left-hand derivative $D_-f(0) = \lim\limits_{h\to 0^-} \dfrac{f(h)-f(0)}{h}$. Does $f'(0)$ exist?

19. $f(x) = \begin{cases} 2x+1 & \text{if } x<0 \\ 3x+1 & \text{if } x\ge 0 \end{cases}$

The left-hand derivative is

$$D_-f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{2h+1-1}{h} = 2$$

The right-hand derivative is

$$D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{3h+1-1}{h} = 3$$

Since the one-sided limits do not agree $(2 \neq 3)$, $f'(0)$ does not exist.

20. $f(x) = \begin{cases} 0 & \text{if } x<0 \\ 2x & \text{if } x\ge 0 \end{cases}$

The left-hand derivative is

$$\mathrm{D}_-f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$$

The right-hand derivative is

$$\mathrm{D}_+f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$$

Since the one-sided limits do not agree $(0 \neq 2)$, $f'(0)$ does not exist.

8	Understand the relationship between continuity and differentiability.	(19-22)	151
	فهم العلاقة بين الاتصال والاشتقاق	32	152

In exercises 19–22, compute the right-hand derivative $D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ and the left-hand derivative $D_-f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$. Does $f'(0)$ exist?

21. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$

The left-hand derivative is

$$D_-f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2-0}{h} = 0$$

The right-hand derivative is

$$D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^3-0}{h} = 0$$

Since the one-sided limits are same $(0 = 0)$, $f'(0)$ exist.

8	Understand the relationship between continuity and differentiability.	(19-22)	151
	فهم العلاقة بين الاتصال والاشتقاق	32	152

In exercises 19–22, compute the right-hand derivative $D_+f(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ and the left-hand derivative $D_-f(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$. Does $f'(0)$ exist?

22. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x^2 + 2x & \text{if } x \geq 0 \end{cases}$

The left-hand derivative is

$$\begin{aligned} D_-f(0) &= \lim_{h\to 0+} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0-} \frac{2h}{h} = 2 \end{aligned}$$

The right-hand derivative is

$$\begin{aligned} D_+f(0) &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0+} \frac{h^2+2h}{h} \\ &= \lim_{h\to 0+} \frac{h(h+2)}{h} \\ &= \lim_{h\to 0+} h+2 = 2 \end{aligned}$$

Since the one-sided limits are same $(2 = 2)$, $f'(0)$ exist.

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8	Understand the relationship between continuity and differentiability.	(19-22)	151
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32. For $f(x) = \begin{cases} x^2 + 2x, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$ find all real numbers a and b such that $f'(0)$ exists.

$$f(x) = \begin{cases} x^2 + 2x & x < 0 \\ ax + b & x \geq 0 \end{cases}$$

For $h < 0$, $f(h) = h^2 + 2h$, $f(0) = b$

$$D_- f(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{h^2 + 2h - b}{h}$$

For f to be differentiable $D_- f(0)$ must exist.

$D_- f(0)$ exists if and only if $b = 0$.

Substituting $b = 0$, we get

$$D_- f(0) = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2$$

For $h > 0$, $f(h) = ah + b$, $f(0) = b$

$$D_+ f(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{ah + b - b}{h}$$

$$= \lim_{h \to 0^+} \frac{ah}{h} = a$$

$D_+ f(0) = 2$ if and only if $a = 2$.

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Question 9

Find the derivative of a function at a given point using the Power Rule

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Exercise 33 - 38

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9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

In exercises 33 and 34, (a) determine the value(s) of x for which the tangent line to $y = f(x)$ is horizontal. (b) Graph the function and determine the graphical significance of each such point. (c) Determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the x-axis at a 45° angle.

33. $f(x) = x^3 - 3x + 1$

(a) $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

The tangent line to $y = f(x)$ is horizontal when

$f'(x) = 0$

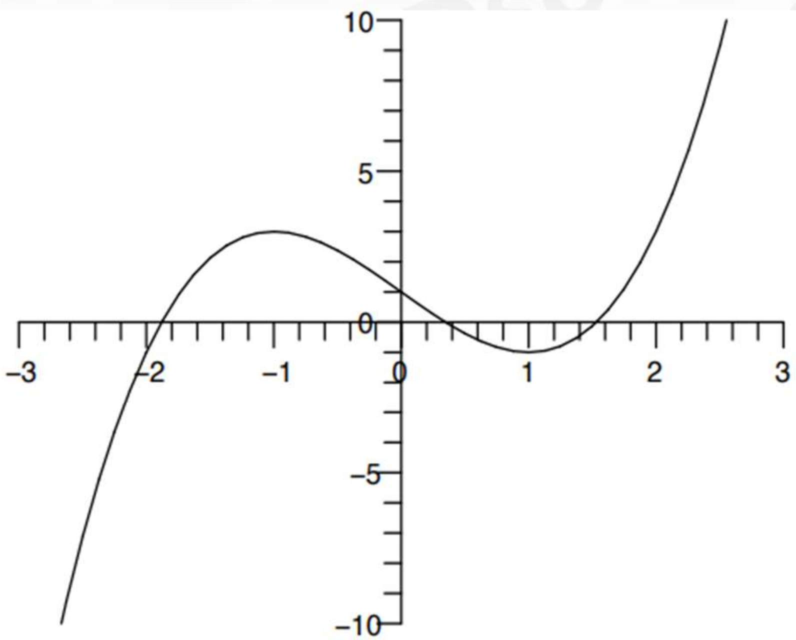
$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow 3(x + 1)(x - 1) = 0$

$x = -1$ or $x = 1.$

(b) The graph shows that the first is a relative maximum, the second is a relative minimum.



(c) Now to determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when

$f'(x) = 1.$

$3x^2 - 3 = 1$

$(x^2 - 1) = \frac{1}{3}$

$x^2 = \frac{4}{3}$

$x = \pm\frac{2}{\sqrt{3}}$

9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

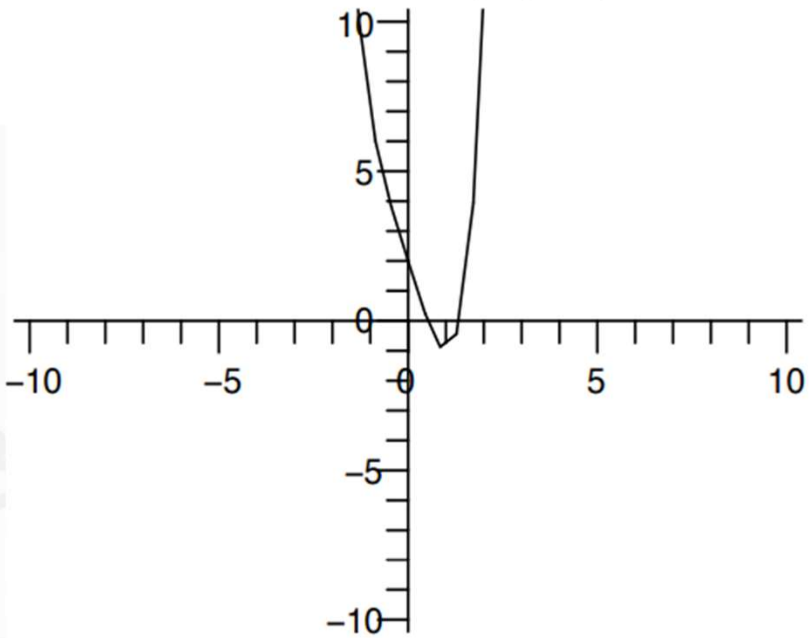
In exercises 33 and 34, (a) determine the value(s) of x for which the tangent line to $y = f(x)$ is horizontal. (b) Graph the function and determine the graphical significance of each such point. (c) Determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the x-axis at a 45° angle.

34. $f(x) = x^4 - 4x + 2$

(a) Now to determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when

$$f'(x) = 1$$
$$3x^2 - 3 = 1$$
$$3(x^2 - 1) = 1$$
$$(x^2 - 1) = \frac{1}{3}$$
$$x^2 = \frac{4}{3}$$
$$x = \pm\frac{2}{\sqrt{3}}$$

(b) The graph shows that the function has global minimum at $(1, -1)$



(c) Now to determine the value (s) of for which the tangent line to $y = f(x)$ intersects the axis at 45^o angle that is when

$$f'(x) = 1$$
$$4x^3 - 4 = 1$$
$$(x^3 - 1) = \frac{1}{4}$$
$$x^3 = \frac{5}{4} = \left(\frac{5}{2}\right)^{1/3}$$

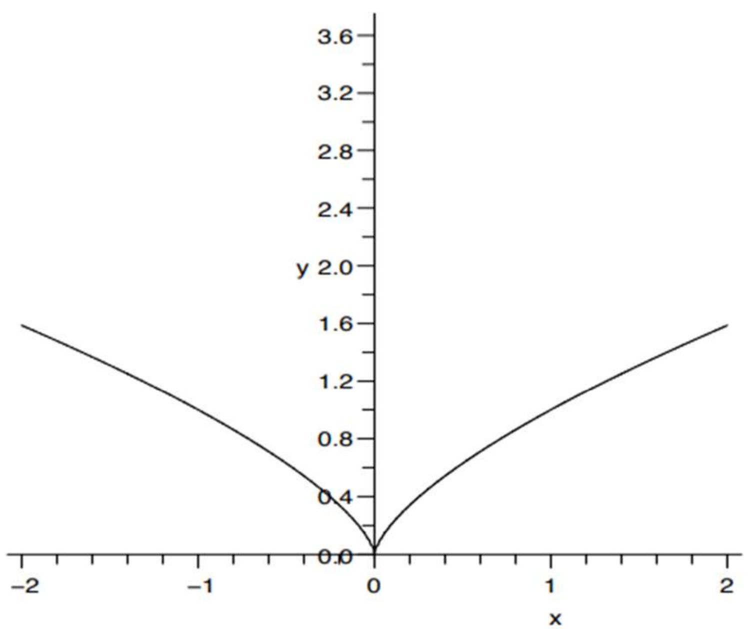
9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

In exercises 35 and 36, (a) determine the value(s) of x for which the slope of the tangent line to $y = f(x)$ does not exist. (b) Graph the function and determine the graphical significance of each such point.

35. (a) $f(x) = x^{2/3}$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

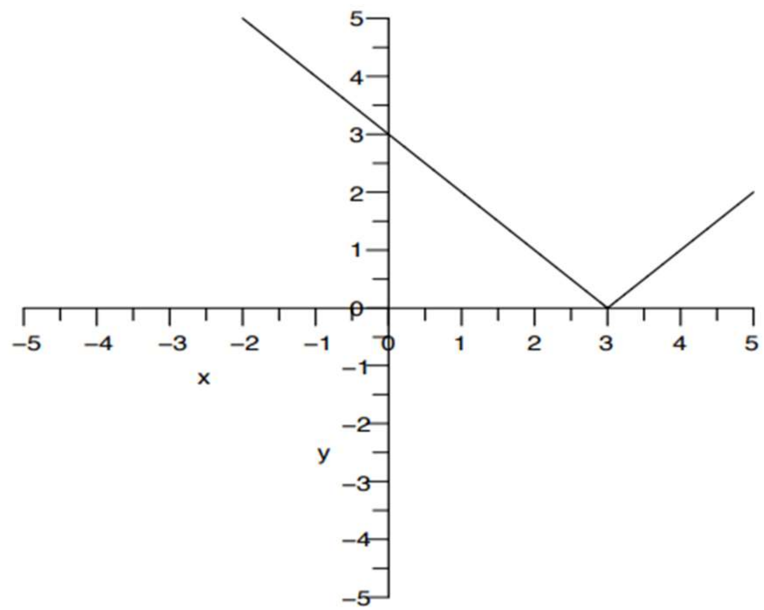
The slope of the tangent line to $y = f(x)$ does not exist where the derivative is undefined, which is only when x=0.



(b) $f(x) = |x - 5|$

$$f'(x) = \begin{cases} 1 \text{ when } x > 3 \\ -1 \text{ when } x < 3 \end{cases}$$

$f'(x)$ is not defined at $x = 3$.

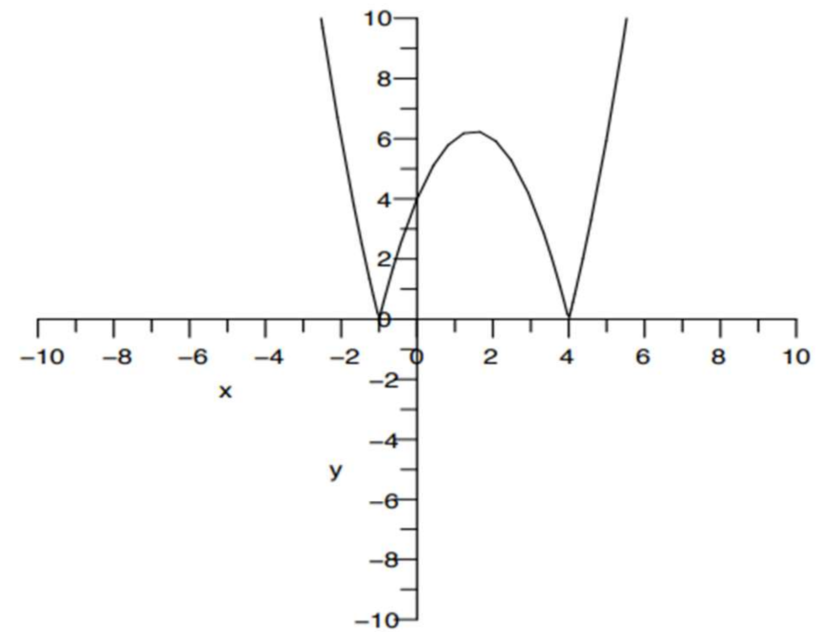


Though the graph of function is continuous at $x = 3$ tangent line does not exist as at this point there is sharp corner.

(c) $f(x) = |x^2 - 3x - 4|$

$$f'(x) = \begin{cases} 2x - 3 \text{ when } x > 4 \text{ or } x < -1 \\ -2x + 3 \text{ when } -1 < x < 4 \end{cases}$$

$f'(x)$ is not defined at $x = -1, 4$.



The graph shows that the function has global minima at $(-1, 0)$ and$(4, 0)$. The function has relative maximum at $\left(\frac{3}{2}, \frac{25}{4}\right)$.

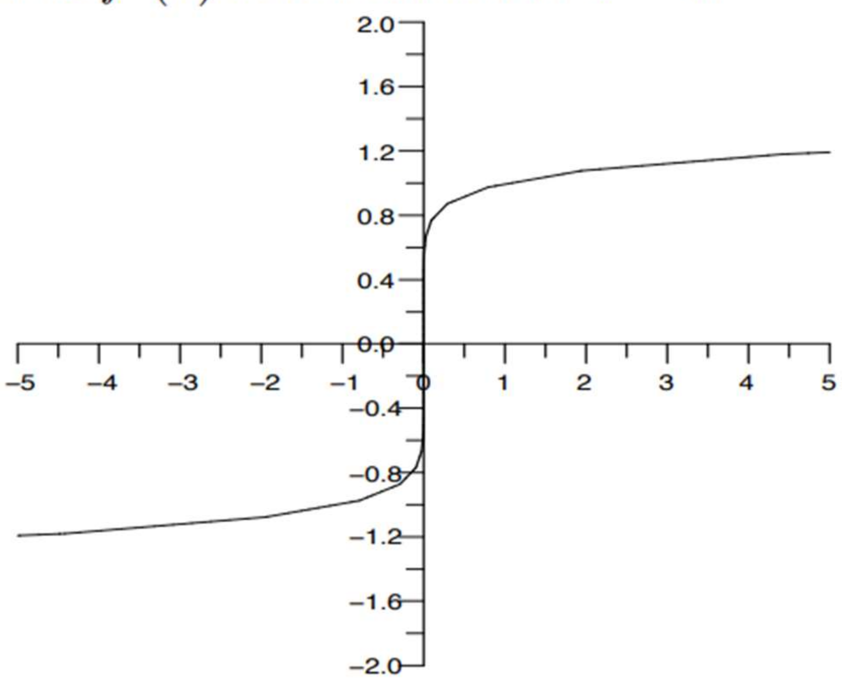
9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

In exercises 35 and 36, (a) determine the value(s) of x for which the slope of the tangent line to $y = f(x)$ does not exist. (b) Graph the function and determine the graphical significance of each such point.

36. (a) $f(x) = x^{1/3}$

$f'(x) = \frac{1}{9}x^{-8/9} = \frac{1}{9x^{8/9}}$

The $f'(x)$ is not defined at $x = 0$.

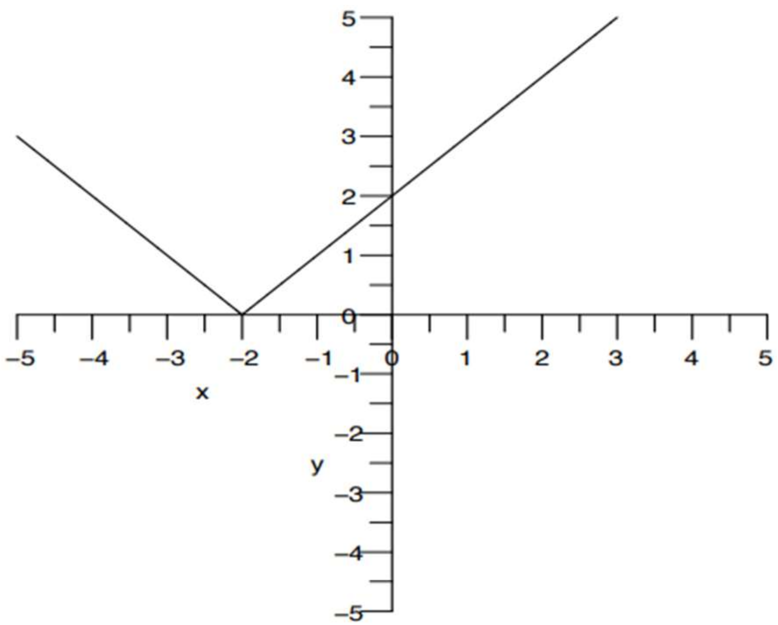


The graphical significance of this point is that there is vertical tangent here.

(b) $f(x) = |x + 2|$

$$f'(x) = \begin{cases} 1 \text{ when } x > -2 \\ -1 \text{ when } x < -2 \end{cases}$$

The $f'(x)$ is not defined at $x = -2$.



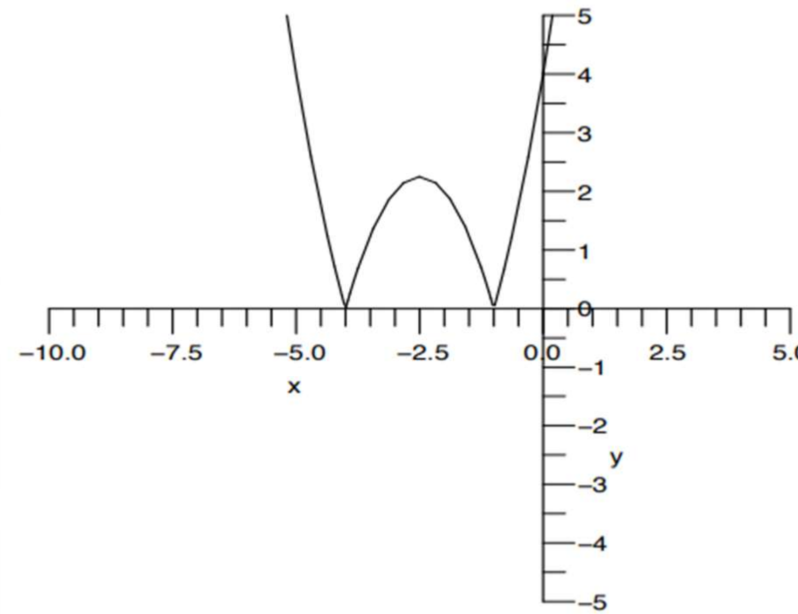
Though the graph of function is continuous at $x = -2$, tangent line does not exist as at this point there is sharp corner.

(c) $f(x) = |x^2 + 5x + 4|$

$f(x) = |x^2 + 5x + 4| = |(x + 4)(x + 1)|$

$$f'(x) = \begin{cases} 2x + 5 \text{ when } x < -4 \text{ or } x > -1 \\ -2x - 5 \text{ when } -4 < x < -1 \end{cases}$$

The $f'(x)$ is not defined at $x = -4, -1$.



The graph shows that the function has global minima at $(-4, 0)$ and $(-1, 0)$. The function has relative maxima at $(-2.5, 2.25)$.

9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

37. Find all values of x for which the tangent line to $y = x^3 - 3x + 1$ is (a) at an angle of 45° with the x-axis; (b) at an angle of 30° with the x-axis, assuming that the angles are measured counterclockwise.

(a) $y = x^3 - 3x + 1$

$y' = 3x^2 - 3 = 3(x^2 - 1)$

The tangent line to $y = f(x)$ intersects the x-axis at a 45° angle when

$f'(x) = 1$

$\Leftrightarrow 3(x^2 - 1) = 1$

$\Leftrightarrow x^2 = 1 + \frac{1}{3}$

$\Leftrightarrow x = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}$

(b) The tangent line to $y = f(x)$ intersects the x-axis at a 30° angle when

$f'(x) = \frac{1}{\sqrt{3}}.$

$\Leftrightarrow 3(x^2 - 1) = \frac{1}{\sqrt{3}}$

$\Leftrightarrow x^2 = 1 + \frac{1}{3\sqrt{3}}$

$\Leftrightarrow x = \left(1 + \frac{1}{3\sqrt{3}}\right)^{1/2}$ or

$x = -\left(1 + \frac{1}{3\sqrt{3}}\right)^{1/2}$

9	Find the derivative of a function at a given point using the Power Rule. إيجاد مشتقة دالة ما باستخدام قاعدة القوة عند نقطة معطاة	(33-38)	161

38. Find all values of x for which the tangent lines to $y = x^3 + 2x + 1$ and $y = x^4 + x^3 + 3$ are (a) parallel; (b) perpendicular.

Answers depend on CAS.

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Question 10

Use differentiation rules and higher derivatives in solving real-life problems

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Exercise 21 - 26

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10	Use differentiation rules and higher derivatives in solving real-life problems. استخدام قواعد الاشتقاق والمشتقات العليا في حل مسائل حياتية	(21-26)	161

In exercises 21–24, use the given position function to find the velocity and acceleration functions.

21. $s(t) = -16t^2 + 40t + 10$

$$v(t) = s'(t) = -32t + 40$$
$$a(t) = v'(t) = s''(t) = -32$$

22. $s(t) = -4.9t^2 + 12t - 3$

$$v(t) = s'(t) = -9.8t + 12$$
$$a(t) = v'(t) = s''(t) = -9.8$$

23. $s(t) = \sqrt{t} + 2t^2$

$$v(t) = s'(t) = \frac{1}{2}t^{-1/2} + 4t$$
$$a(t) = v'(t) = s''(t) = -\frac{1}{4}t^{-3/2} + 4$$

24. $s(t) = 10 - \frac{10}{t}$

$$v(t) = s'(t) = 10t^{-2}$$
$$a(t) = s''(t) = -20t^{-3}$$

In exercises 25 and 26, the given function represents the height of an object. Compute the velocity and acceleration at time $t = t_0$. Is the object going up or down?

25. $h(t) = -16t^2 + 40t + 5$, (a) $t_0 = 1$ (b) $t_0 = 2$

$$h(t) = -16t^2 + 40t + 5$$
$$v(t) = h'(t) = -32t + 40$$
$$a(t) = v'(t) = h''(t) = -32$$

(a) At time $t_0 = 1$
$v(1) = 8$, object is going up.
$a(1) = -32$, speed is decreasing.

(b) At time $t_0 = 2$
$v(2) = -24$, object is going down.
$a(2) = -32$, speed is increasing.

26. $h(t) = 10t^2 - 24t$, (a) $t_0 = 2$ (b) $t_0 = 1$

$$h(t) = 10t^2 - 24t$$
$$v(t) = h'(t) = 20t - 24$$
$$a(t) = v'(t) = h''(t) = 20$$

(a) At time $t_0 = 2$
$v(2) = 16$, object is going up.
$a(2) = 20$, speed is increasing.

(b) At time $t_0 = 1$
$v(1) = -4$, object is going down.
$a(1) = 20$, speed is decreasing.

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Question 11

Apply the Quotient Rule to find derivatives

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Exercise 5 – 12
Exercise 19, 20, 22, 24

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 1–16, find the derivative of each function.

5. $g(t) = \dfrac{3t-2}{5t+1}$

$$g(t) = \frac{3t-2}{5t+1}$$

$$g'(t) = \frac{\left((5t+1)\frac{d}{dt}(3t-2)\right) - \left((3t-2)\frac{d}{dt}(5t+1)\right)}{(5t+1)^2}$$

$$= \frac{3(5t+1) - 5(3t-2)}{(5t+1)^2}$$

$$= \frac{15t+3-15t+10}{(5t+1)^2} = \frac{13}{(5t+1)^2}$$

6. $g(t) = \dfrac{t^2+2t+5}{t^2-5t+1}$

$$g(t) = \frac{t^2+2t+5}{t^2-5t+1}$$

$$g'(t) = \frac{\left((t^2-5t+1)\frac{d}{dt}(t^2+2t+5)\right) - \left((t^2+2t+5)\frac{d}{dt}(t^2-5t+1)\right)}{(t^2-5t+1)^2}$$

$$= \frac{(t^2-5t+1)(2t+2) - (t^2+2t+5)(2t-5)}{(t^2-5t+1)^2}$$

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 1–16, find the derivative of each function.

7. $f(x) = \dfrac{3x - 6\sqrt{x}}{5x^2 - 2}$

$$f(x) = \frac{3x - 6\sqrt{x}}{5x^2 - 2} = \frac{3(x - 2x^{1/2})}{5x^2 - 2}$$

$$f'(x) = 3\frac{\left((5x^2-2)\frac{d}{dx}(x-2x^{1/2}) - (x-2x^{1/2})\frac{d}{dx}(5x^2-2)\right)}{(5x^2-2)^2}$$

$$= 3\frac{\left((5x^2-2)(1-x^{-1/2}) - (x-2x^{1/2})(10x)\right)}{(5x^2-2)^2}$$

8. $f(x) = \dfrac{6x - 2/x}{x^2 + \sqrt{x}}$

$$f(x) = \frac{6x - 2x^{-1}}{x^2 + x^{1/2}}$$

$$f'(x) = \frac{(x^2+x^{1/2})\frac{d}{dx}(6x-2x^{-1}) - (6x-2x^{-1})\frac{d}{dx}(x^2+x^{1/2})}{(x^2+x^{1/2})^2}$$

$$= \frac{(x^2+x^{1/2})(6+2x^{-2}) - (6x-2x^{-1})(2x+\frac{1}{2}x^{-1/2})}{(x^2+x^{1/2})^2}$$

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 1–16, find the derivative of each function.

9. $f(u) = \dfrac{(u+1)(u-2)}{u^2-5u+1}$

$$f(u) = \frac{(u+1)(u-2)}{u^2-5u+1} = \frac{u^2-u-2}{u^2-5u+1}$$

$$f'(u) = \frac{\left((u^2-5u+1)\frac{d}{du}(u^2-u-2)\right)-\left((u^2-u-2)\frac{d}{du}(u^2-5u+1)\right)}{(u^2-5u+1)^2}$$

$$= \frac{(u^2-5u+1)(2u-1)-(u^2-u-2)(2u-5)}{(u^2-5u+1)^2}$$

$$= \frac{2u^3-10u^2+2u-u^2+5u-1-2u^3+2u^2+4u+5u^2-5u-10}{(u^2-5u+1)^2}$$

$$= \frac{-4u^2+6u-11}{(u^2-5u+1)^2}$$

10. $f(u) = \dfrac{2u}{u^2+1}(u+3)$

$$f(u) = \frac{(2u)(u+3)}{u^2+1} = \frac{2u^2+6u}{u^2+1}$$

$$f'(u) = \frac{\left((u^2+1)\frac{d}{du}(2u^2+6u)\right)-\left((2u^2+6u)\frac{d}{du}(u^2+1)\right)}{(u^2+1)^2}$$

$$= \frac{(u^2+1)(4u+6)-(2u^2+6u)(2u)}{(u^2+1)^2}$$

$$= \frac{4u^3+6u^2+4u+6-4u^3-12u^2}{(u^2+1)^2}$$

$$= \frac{-6u^2+4u+6}{(u^2+1)^2}$$

$$= \frac{2(-3u^2+2u+3)}{(u^2+1)^2}$$

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 1–16, find the derivative of each function.

11. $f(x) = \dfrac{x^2 + 3x - 2}{\sqrt{x}}$

We do not recommend treating this one as a quotient, but advise preliminary simplification.

$$f(x) = \frac{x^2 + 3x - 2}{\sqrt{x}}$$

$$= \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}}$$

$$= x^{3/2} + 3x^{1/2} - 2x^{-1/2}$$

$$f'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}$$

12. $f(x) = \dfrac{x^2 - 2x}{x^2 + 5x}$

$$f(x) = \frac{x^2 - 2x}{x^2 + 5x}$$

$$f'(u) = \frac{(x^2 + 5x)\frac{d}{dx}(x^2 - 2x) - (x^2 - 2x)\frac{d}{dx}(x^2 + 5x)}{(x^2 + 5x)^2}$$

$$= \frac{(x^2 + 5x)(2x - 2) - (x^2 - 2x)(2x + 5)}{(x^2 + 5x)^2}$$

In exercises 17–20, find an equation of the tangent line to the graph of $y = f(x)$ at $x = a$.

19. $f(x) = \dfrac{x+1}{x+2}, a = 0$

$$f(x) = \frac{x+1}{x+2}$$

By The Quotient Rule, we have

$$f'(x) = \frac{\left((x+2)\frac{d}{dx}(x+1)\right) - \left((x+1)\frac{d}{dx}(x+2)\right)}{(x+2)^2}$$

$$= \frac{(x+2)-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

At $x = a = 0$,

$$f(0) = \frac{0+1}{0+2} = \frac{1}{2}$$

$$f'(0) = \frac{1}{4}.$$

The line with slope $\frac{1}{4}$ and passing through the point $\left(0, \frac{1}{2}\right)$ has equation $y = \frac{1}{4}x + \frac{1}{2}$.

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 17–20, find an equation of the tangent line to the graph of $y = f(x)$ at $x = a$.

20. $f(x) = \dfrac{x+3}{x^2+1}, a = 1$

By The Quotient Rule, we have

$$f'(x) = \frac{\left((x^2+1)\frac{d}{dx}(x+3)\right) - \left((x+3)\frac{d}{dx}(x^2+1)\right)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - (x+3)(2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - (2x^2+6x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2-6x}{(x^2+1)^2}$$

$$= \frac{-x^2-6x+1}{(x^2+1)^2}.$$

At $x = a = 1$,

$f(1) = \frac{1+3}{1^2+1} = 2$

$$f'(1) = \frac{-1-6+1}{(1+1)^2} = -\frac{6}{4} = -\frac{3}{2}.$$

The line with slope $-\dfrac{3}{2}$ and passing through the point $(1, 2)$ has equation $y = -\dfrac{3}{2}(x-1)+2$.

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 21–24, assume that f and g are differentiable with $f(0) = -1$, $f(1) = -2$, $f'(0) = -1$, $f'(1) = 3$, $g(0) = 3$, $g(1) = 1$, $g'(0) = -1$ and $g'(1) = -2$. Find an equation of the tangent line to the graph of $y = h(x)$ at $x = a$.

22. $h(x) = \dfrac{f(x)}{g(x)}$; (a) $a = 1$; (b) $a = 0$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

(a) At $x = a = 1$,

$$h(1) = \frac{f(1)}{g(1)} = -\frac{2}{1} = -2$$

$$h'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{(g(1))^2} = \frac{(1)(3) - (-2)(-2)}{1^2} = \frac{3-4}{1} = -1.$$

So, the equation of the tangent line is $y = -(x-1) - 2$.

(b) At $x = a = 0$,

$$h(0) = \frac{f(0)}{g(0)} = -\frac{1}{3}$$

$$h'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2} = \frac{(-1)(3) - (-1)(-1)}{(3)^2} = \frac{-3-1}{9} = -\frac{4}{9}.$$

So, the equation of the tangent line is $y = -\frac{4}{9}x - \frac{1}{3}$.

11	Apply the Quotient Rule to find derivatives.	(5-12)	169
	تطبيق قاعدة مشتقة خارج قسمة دالتين	(19,20,22,24)	

In exercises 21–24, assume that f and g are differentiable with $f(0) = -1$, $f(1) = -2$, $f'(0) = -1$, $f'(1) = 3$, $g(0) = 3$, $g(1) = 1$, $g'(0) = -1$ and $g'(1) = -2$. Find an equation of the tangent line to the graph of $y = h(x)$ at $x = a$.

24. $h(x) = \dfrac{x^2}{g(x)}$; (a) $a = 1$; (b) $a = 0$

$$h'(x) = \frac{2xg(x) - x^2 g'(x)}{(g(x))^2}$$

(a) At $x = a = 1$,

$$h(1) = \frac{1^2}{g(1)} = \frac{1}{1} = 1$$

$$h'(1) = \frac{2 \times 1 \times g(1) - 1^2 g'(1)}{(g(1))^2}$$

$$= \frac{(2)(1)(1) - (1)(-2)}{1^2}$$

$$= \frac{2+2}{1} = 4.$$

So, the equation of tangent line is $y = 4(x-1) + 1$.

(b) At $x = a = 0$,

$$h(0) = \frac{0^2}{g(0)} = \frac{0}{3} = 0$$

$$h'(0) = \frac{2 \times 0 \times g(0) - 0^2 g'(0)}{(g(0))^2} = 0.$$

So, the equation of the tangent line is $y = 0$.

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Question 12

Find the derivative of an inverse function using the Chain Rule

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Exercise 17 - 22

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Justin Dsouza
Teacher

12	Find the derivative of an inverse function using the Chain Rule. إيجاد مشتقة معكوس دالة باستخدام قاعدة السلسلة	(17-22)	176

In exercises 17–22, f has an inverse g. Use Theorem 5.2 to find $g'(a)$.

17. $f(x) = x^3 + 4x - 1, a = -1$

$f(x) = x^3 + 4x - 1$ is a one-to-one function with $f(0) = -1$ and $f'(0) = 4$. Therefore $g(-1) = 0$ and

$$g'(-1) = \frac{1}{f'(g(-1))} = \frac{1}{f'(0)} = \frac{1}{4}.$$

18. $f(x) = x^5 + 4x - 2, a = -2$

$f(x) = x^5 + 4x - 2$ is a one-to-one function with $f(0) = -2$ and $f'(0) = 4$. Therefore $g(-2) = 0$ and

$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(0)} = \frac{1}{4}$$

19. $f(x) = x^5 + 3x^3 + x, a = 5$

$f(x) = x^5 + 3x^3 + x$ is a one-to-one function with $f(1) = 5$ and $f'(1) = 5 + 9 + 1 = 15$. Therefore $g(5) = 1$ and

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{15}.$$

20. $f(x) = x^3 + 2x + 1, a = -2$

$f(x) = x^3 + 2x + 1$ is a one-to-one function with $f(-1) = -2$ and $f'(-1) = 5$. Therefore $g(-2) = -1$ and

$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(-1)} = \frac{1}{5}.$$

12	Find the derivative of an inverse function using the Chain Rule. إيجاد مشتقة معكوس دالة باستخدام قاعدة السلسلة	(17-22)	176

In exercises 17–22, f has an inverse g. Use Theorem 5.2 to find $g'(a)$.

21. $f(x) = \sqrt{x^3 + 2x + 4}, a = 2$

$f(x) = \sqrt{x^3 + 2x + 4}$ is a one-to-one function and $f(0) = 2$ so $g(2) = 0$. Meanwhile,

$$f'(x) = \frac{1}{2\sqrt{x^3 + 2x + 4}}(3x^2 + 2)$$

$$f'(0) = 1/2$$

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(0)} = 2.$$

22. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}, a = 3$

$f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$ is a one-to-one function and $f(1) = 3$ so $g(3) = 1$. Meanwhile,

$$f'(x) = \frac{5x^4 + 12x^2 + 3}{2\sqrt{x^5 + 4x^3 + 3x + 1}}$$

$$f'(1) = \frac{20}{6} = \frac{10}{3}$$

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{3}{10}.$$

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Question 13

Find the derivatives of trigonometric functions using differentiation rules

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Exercise 1 - 22

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Justin Dsouza
Teacher

13	Find the derivatives of trigonometric functions using differentiation rules. إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق	(1-22)	184

In exercises 1–18, find the derivative of each function.

1. $f(x) = 4\sin 3x - x$

$$\begin{aligned} f(x) &= 4\sin 3x - x \\ f'(x) &= 4(\cos 3x)(3) - 1 \\ &= 12\cos 3x - 1 \end{aligned}$$

2. $f(x) = 4x^2 - 3\tan 2x$

$$\begin{aligned} f(x) &= 4x^2 - 3\tan 2x \\ f'(x) &= 4(2x) - 3\sec^2(2x)(2) \\ &= 8x - 6\sec^2(2x) \end{aligned}$$

3. $f(t) = \tan^3 2t - \csc^4 3t$

$$\begin{aligned} f(t) &= \tan^3 2t - \csc^4 3t \\ f'(t) &= 3\tan^2(2t)\sec^2(2t)(2) \\ &\quad - 4\csc^3(3t)[-\csc(3t)\cot(3t)](3) \\ &= 6\tan^2(2t)\sec^2(2t) \\ &\quad + 12\csc^4(3t)\cot(3t) \end{aligned}$$

4. $f(t) = t^2 + 2\cos^2 4t$

$$\begin{aligned} f(t) &= t^2 + 2\cos^2 4t \\ f'(t) &= 2t + 4\cos(4t)[-\sin(4t)](4) \\ &= 2t - 16\sin(4t)\cos(4t) \end{aligned}$$

In exercises 1–18, find the derivative of each function.

5. $f(x) = x\cos 5x^2$

$$f(x) = x\cos 5x^2$$
$$f'(x) = (1)\cos 5x^2 + x(-\sin 5x^2)\cdot 10x$$
$$= \cos 5x^2 - 10x^2 \sin 5x^2$$

6. $f(x) = x^2 \sec 4x$

$$f(x) = x^2 \sec 4x$$
$$f'(x) = x^2(\sec 4x \tan 4x)4 + (\sec 4x)2x$$
$$= 4x^2(\sec 4x \tan 4x) + 2x\sec(4x)$$

7. $f(x) = \dfrac{\sin x^2}{x^2}$

$$f(x) = \frac{\sin(x^2)}{x^2}$$
$$f'(x) = \frac{x^2\cos(x^2)\cdot 2x - \sin(x^2)\cdot 2x}{x^4}$$
$$= \frac{2x[x^2\cos(x^2) - \sin(x^2)]}{x^4}$$
$$= \frac{2[x^2\cos(x^2) - \sin(x^2)]}{x^3}$$

8. $f(x) = \dfrac{x^2}{\csc^4 2x}$

$$f'(x) = \frac{2x[\csc^4(2x)] - 4x^2[\csc^3(2x)][-\csc(2x)\cot(2x)](2)}{[\csc^4(2x)]^2}$$
$$= \frac{2x}{\csc^4(2x)} + \frac{8x^2[\csc^4(2x)\cot(2x)]}{[\csc^4(2x)]^2}$$
$$= \frac{2x}{\csc^4(2x)} + \frac{8x^2\cot(2x)}{\csc^4(2x)}$$
$$= \frac{2x + 8x^2\cot(2x)}{\csc^4(2x)}$$

13	Find the derivatives of trigonometric functions using differentiation rules. إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق	(1-22)	184

In exercises 1–18, find the derivative of each function.

9. $f(t) = \sin 3t \sec 3t$

$$f(t) = \sin 3t \sec 3t = \tan 3t$$
$$f'(t) = \frac{d}{dt}[\tan(3t)] = \sec^2(3t)(3)$$
$$= 3\sec^2(3t)$$

10. $f(t) = \sqrt{\cos 5t \sec 5t}$

$$f(t) = \sqrt{\cos 5t \sec 5t}$$
$$= \sqrt{\cos 5t \cdot \frac{1}{(\cos 5t)}} = 1$$
$$f'(t) = \frac{d}{dt}(1) = 0$$

11. $f(w) = \dfrac{1}{\sin 4w}$

$$f(w) = \frac{1}{\sin 4w}$$
$$f'(w) = \frac{-1}{(\sin 4w)^2}\cos 4w\,(4)$$
$$= \frac{-4\cos 4w}{\sin^2 4w}$$

12. $f(w) = w^2 \sec^2 3w$

$$f(w) = w^2\sec^2 3w$$
$$f'(w) = w^2(2\sec 3w)(\sec 3w \tan 3w)(3)$$
$$+ \sec^2(3w)(2w)$$
$$= 6w^2\sec^2 3x \tan 3w + 2w\sec^2 3w$$

13	Find the derivatives of trigonometric functions using differentiation rules. إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق	(1-22)	184

In exercises 1–18, find the derivative of each function.

13. $f(x) = 2\sin 2x \cos 2x$

$$
\begin{aligned}
f(x) &= 2\sin(2x)\cos(2x) \\
f'(x) &= 2\{\sin(2x)[-\sin(2x)](2) \\
&\quad + \cos(2x)[\cos(2x)](2)\} \\
&= -4\sin^2(2x) + 4\cos^2(2x) \\
&= 4\cos^2(2x) - 4\sin^2(2x)
\end{aligned}
$$

14. $f(x) = 4\sin^2 3x + 4\cos^2 3x$

$$
\begin{aligned}
f(x) &= 4\sin^2(3x) + 4\cos^2(3x) \\
&= 4\left[\sin^2(3x) + \cos^2(3x)\right] = 4 \\
f'(x) &= \frac{d}{dx}(4) = 0
\end{aligned}
$$

15. $f(x) = \tan\sqrt{x^2+1}$

$$
\begin{aligned}
f(x) &= \tan\sqrt{x^2+1} \\
f'(x) &= \left(\sec^2\sqrt{x^2+1}\right) \\
&\quad \cdot \left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) \\
&= \frac{x}{\sqrt{x^2+1}}\sec^2\sqrt{x^2+1}
\end{aligned}
$$

16. $f(x) = 4x^2 \sin x \sec 3x$

$$
\begin{aligned}
f(x) &= 4x^2\sin x\sec 3x \\
f'(x) &= 8x\sin x\sec 3x + 4x^2[\cos x\sec 3x \\
&\quad + \sin x\sec 3x\tan 3x(3)]
\end{aligned}
$$

13	Find the derivatives of trigonometric functions using differentiation rules.	(1-22)	184
	إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق		

In exercises 1–18, find the derivative of each function.

17. $f(x) = \sin^3\left(\cos\sqrt{x^3+2x^2}\right)$

$$f'(x) = 3\sin^2\left(\cos\sqrt{x^3+2x^2}\right) \cdot \cos\left(\cos\sqrt{x^3+2x^2}\right) \cdot \left(-\sin\sqrt{x^3+2x^2}\right) \cdot \frac{1}{2}\left(x^3+2x^2\right)^{-1/2}\left(3x^2+4x\right)$$

$$= \frac{3}{2}\left(3x^2+4x\right)\left(x^3+2x^2\right)^{-1/2} \cdot \sin^2\left(\cos\sqrt{x^3+2x^2}\right) \cdot \cos\left(\cos\sqrt{x^3+2x^2}\right) \cdot \left(-\sin\sqrt{x^3+2x^2}\right)$$

18. $f(x) = \tan^4(\sin^2(x^3+2x))$

$$f(x) = \tan^4\left[\sin^2\left(x^3+2x\right)\right]$$

$$f'(x) = 4\left[\tan^3\left(\sin^2\left(x^3+2x\right)\right)\right] \cdot \left[\sec^2\left(\sin^2\left(x^3+2x\right)\right)\right] \cdot \left[2\sin\left(x^3+2x\right)\right] \cdot \left[\cos\left(x^3+2x\right)\right] \cdot \left(3x^2+2\right)$$

13	Find the derivatives of trigonometric functions using differentiation rules. إيجاد مشتقات الدوال المثلثية باستخدام قواعد الاشتقاق	(1-22)	184

In exercises 19–22, find the derivative of each function.

19. (a) $f(x) = \sin x^2$

(b) $f(x) = \sin^2 x$

(c) $f(x) = \sin 2x$

(a) $f(x) = \sin x^2$

$f'(x) = \cos(x^2) \cdot (2x) = 2x\cos(x^2)$

(b) $f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x$

(c) $f(x) = \sin 2x$

$f'(x) = \cos 2x\,(2) = 2\cos 2x$

20. (a) $f(x) = \cos\sqrt{x}$

(b) $f(x) = \sqrt{\cos x}$

(c) $f(x) = \cos \frac{1}{2}x$

(a) $f(x) = \cos\sqrt{x}$

$$f'(x) = \left(-\sin\sqrt{x}\right) . \frac{1}{2}(x)^{-1/2}$$

$$= -\frac{1}{2}(x)^{-1/2}\sin\sqrt{x}$$

(b) $f(x) = \sqrt{\cos x}$

$$f'(x) = \frac{1}{2}(\cos x)^{-1/2} . (-\sin x)$$

$$= -\frac{1}{2}\sin x(\cos x)^{-1/2}$$

(c) $f(x) = \cos\left(\frac{1}{2}x\right)$

$$f'(x) = -\sin\left(\frac{1}{2}x\right) . \left(\frac{1}{2}\right)$$

$$= -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$$

In exercises 19–22, find the derivative of each function.

21. (a) $f(x) = \sin x^2 \tan x$

(b) $f(x) = \sin^2(\tan x)$

(c) $f(x) = \sin(\tan^2 x)$

22. (a) $f(x) = \sec x^2 \tan x^2$

(b) $f(x) = \sec^2(\tan x)$

(c) $f(x) = \sec(\tan^2 x)$

(a) $f(x) = \sin x^2 \tan x$

$$f'(x) = \sin x^2 \left(\sec^2 x\right) + 2x \cos x^2 \tan x$$

(b) $f(x) = \sin^2(\tan x)$

$$f'(x) = 2\sin(\tan x) \cdot \cos(\tan x) \cdot \sec^2 x$$

(c) $f(x) = \sin\left(\tan^2 x\right)$

$$f'(x) = \left[\cos\left(\tan^2 x\right)\right](2\tan x)\left(\sec^2 x\right)$$
$$= (2\tan x)\left(\sec^2 x\right)\left[\cos\left(\tan^2 x\right)\right]$$

(a) $f(x) = \sec x^2 \tan x^2$

$$f'(x) = \sec^3\left(x^2\right)(2x) + \tan^2\left(x^2\right)\sec\left(x^2\right)(2x)$$
$$= 2x \sec x^2 \left[\sec^2 x^2 + \tan^2 x^2\right]$$

(b) $f(x) = \sec^2(\tan x)$

$$f'(x) = 2\sec(\tan x)\left[\sec(\tan x) . \tan(\tan x)\right]\left(\sec^2 x\right)$$

(c) $f(x) = \sec\left(\tan^2 x\right)$

$$f'(x) = \left[\sec\left(\tan^2 x\right)\tan\left(\tan^2 x\right)\right] \cdot (2\tan x)\left(\sec^2 x\right)$$
$$= \left(2\tan x \sec^2 x\right) \cdot \left[\sec\left(\tan^2 x\right)\tan\left(\tan^2 x\right)\right]$$

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Question 14

Find derivatives of natural logarithmic functions

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Exercise 7, 8 & 22

Exercise 26, 39 - 44

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14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 1–24, differentiate each function.

7. $h(x) = (1/3)^{x^2}$

$$h(x) = \left(\tfrac{1}{3}\right)^{x^2}$$

$$h'(x) = \ln\left(\tfrac{1}{3}\right) \cdot 2x \cdot \left(\tfrac{1}{3}\right)^{x^2}$$

$$= 2x \cdot \ln\left(\tfrac{1}{3}\right) \cdot \left(\tfrac{1}{3}\right)^{x^2}$$

$$= -2x \cdot \ln(3) \cdot \left(\tfrac{1}{3}\right)^{x^2}$$

8. $h(x) = 4^{-x^2}$

$$h(x) = 4^{-x^2}$$

$$h'(x) = 4^{-x^2} \cdot \ln(4) \cdot (-2x)$$

$$= -2x \cdot 4^{-x^2} \cdot \ln(4)$$

14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 1–24, differentiate each function.

22. (a) $h(x) = 2^{e^x}$

$$h'(x) = 2^{e^x} \cdot e^x \cdot \ln 2$$

(b) $f(x) = \dfrac{e^x}{2^x}$

$$f'(x) = \frac{2^x \cdot e^x - e^x \cdot 2^x \cdot \ln 2}{(2^x)^2}$$

$$= \frac{e^x(1 - \ln 2)}{2^x}$$

14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 25–28, find an equation of the tangent line to $y = f(x)$ at $x = 1$.

26. $f(x) = 3^{x^e}$

$$f(x) = 3^{x^e}$$
$$f(1) = 3^{1^e} = 3$$
$$f'(x) = 3^{x^e} \ln 3 \cdot ex^{(e-1)}$$
$$f'(1) = 3 \ln 3 \cdot e$$

So, the equation of the tangent line is,

$$y = 3 \ln 3 \cdot e(x - 1) + 3.$$

14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 39–44, use logarithmic differentiation to find the derivative.

39. $f(x) = x^{\sin x}$

$$f(x) = x^{\sin x}$$
$$\ln f(x) = \sin x . \ln x$$
$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\sin x . \ln x)$$
$$= \cos x . \ln x + \frac{\sin x}{x}$$
$$f'(x) = x^{\sin x}\left(\frac{x \cos x . \ln x + \sin x}{x}\right)$$

40. $f(x) = x^{4-x^2}$

$$f(x) = x^{4-x^2}$$
$$\ln f(x) = (4 - x^2)\ln x$$
$$\frac{f'(x)}{f(x)} = -2x \ln x + (4 - x^2)\frac{1}{x}$$
$$f'(x) = x^{4-x^2}\left(-2x \ln x + (4 - x^2)\frac{1}{x}\right)$$

14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 39–44, use logarithmic differentiation to find the derivative.

41. $f(x) = (\sin x)^x$

$$f(x) = (\sin x)^x$$
$$\ln f(x) = x.\ln(\sin x)$$
$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(x.\ln(\sin x))$$
$$= \frac{x\cos x}{\sin x} + \ln(\sin x)$$
$$= x\cot x + \ln(\sin x)$$

$$f'(x) = (\sin x)^x.(x\cot x + \ln(\sin x))$$

42. $f(x) = (x^2)^{4x}$

$$f(x) = \left(x^2\right)^{4x}$$
$$\ln f(x) = 8x\ln x$$
$$\frac{f'(x)}{f(x)} = 8\ln x + 8x\frac{1}{x}$$
$$f'(x) = \left(x^2\right)^{4x}(8\ln x + 8)$$

14	Find derivatives of natural logarithmic functions.	(7,8,22)	193
	إيجاد مشتقات الدوال اللوغاريتمية الطبيعية	(26,39-44)	194

In exercises 39–44, use logarithmic differentiation to find the derivative.

43. $f(x) = x^{\ln x}$

$$f(x) = x^{\ln x}$$
$$\ln f(x) = \ln x . \ln x = \ln^2 x$$
$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\ln^2 x\right) = \frac{2\ln x}{x}$$
$$f'(x) = x^{\ln x}\left[\frac{2\ln x}{x}\right] = 2x^{[(\ln x)-1]\ln x}$$

44. $f(x) = x^{\sqrt{x}}$

$$f(x) = x^{\sqrt{x}}$$
$$\ln f(x) = \sqrt{x}\ln x$$
$$\frac{f'(x)}{f(x)} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}$$
$$f'(x) = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right)$$

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UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

Question 15

Use implicit differentiation to find derivatives of inverse trigonometric functions

Page 204
Exercise 29 - 34

المدرسة الإماراتية
مدرسة العروبة للتعليم الثانوي
Justin Dsouza
Teacher

15	Use implicit differentiation to find derivatives of inverse trigonometric functions. استخدام الاشتقاق الضمني في إيجاد مشتقات الدوال المثلثية العكسية	(29-34)	204

In exercises 29–34, find the derivative of the given function.

29. (a) $f(x) = \sin^{-1}(x^3+1)$ (b) $f(x) = \sin^{-1}(\sqrt{x})$

$f(x) = \sin^{-1}\left(x^3+1\right)$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\sin^{-1}\left(x^3+1\right)\right].$$

By the Chain rule we get,

$$f'(x) = \frac{1}{\sqrt{1-\left(x^3+1\right)^2}}\frac{d}{dx}\left(x^3+1\right)$$

$$= \frac{1}{\sqrt{1\text{-}(x^3+1)^2}}\left(3x^2\right)$$

$$= \frac{3x^2}{\sqrt{1-\left(x^3+1\right)^2}}.$$

$f(x) = \sin^{-1}\left(\sqrt{x}\right)$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\sin^{-1}\left(\sqrt{x}\right)\right].$$

By the Chain rule, we get

$$f'(x) = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x\left(1-x\right)}}$$

In exercises 29–34, find the derivative of the given function.

30. (a) $f(x) = \cos^{-1}(x^2 + x)$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\cos^{-1}\left(x^2+x\right)\right].$$

By using Chain rule,

$$f'(x) = \frac{-1}{\sqrt{1-\left(x^2+x\right)^2}}\frac{d}{dx}\left(x^2+x\right)$$

$$= \frac{-\left(2x+1\right)}{\sqrt{1-\left(x^2+x\right)^2}}$$

(b) $f(x) = \cos^{-1}(2/x)$

$$f(x) = \cos^{-1}\left(\frac{2}{x}\right)$$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\cos^{-1}\left(\frac{2}{x}\right)\right]$$

By using Chain rule,

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}\frac{d}{dx}\left(\frac{2}{x}\right)$$

$$= \frac{-1}{\sqrt{1-\left(\frac{4}{x^2}\right)}}\left(\frac{-2}{x^2}\right)$$

$$= \frac{2}{x\sqrt{x^2-4}}$$

In exercises 29–34, find the derivative of the given function.

31. (a) $f(x) = \tan^{-1}(\sqrt{x})$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\tan^{-1}(\sqrt{x})\right].$$

By the Chain rule,

$$f'(x) = \frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{(1+x)}\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}(1+x)}$$

(b) $f(x) = \tan^{-1}(1/x)$

$$f(x) = \tan^{-1}\left(\frac{1}{x}\right)$$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right].$$

By the Chain rule,

$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{\left(1+\frac{1}{x^2}\right)}\left(\frac{-1}{x^2}\right)$$

$$= \frac{-1}{(x^2+1)}$$

In exercises 29–34, find the derivative of the given function.

32. (a) $f(x) = \sqrt{2+\tan^{-1} x}$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left(\sqrt{2+\tan^{-1}x}\right).$$

By the Chain rule,

$$f'(x) = \frac{1}{2\sqrt{2+\tan^{-1}x}}\frac{d}{dx}\left(2+\tan^{-1}x\right)$$

$$= \frac{1}{2\sqrt{2+\tan^{-1}x}}\left(\frac{1}{1+x^2}\right)$$

$$= \frac{1}{2\left(1+x^2\right)\sqrt{2+\tan^{-1}x}}$$

(b) $f(x) = e^{\tan^{-1} x}$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left(e^{\tan^{-1}x}\right).$$

By the Chain rule,

$$f'(x) = \left(e^{\tan^{-1}x}\right)\left(\frac{1}{1+x^2}\right)$$

$$= \frac{e^{\tan^{-1}x}}{1+x^2}$$

15	Use implicit differentiation to find derivatives of inverse trigonometric functions. استخدام الاشتقاق الضمني في إيجاد مشتقات الدوال المثلثية العكسية	(29-34)	204

In exercises 29–34, find the derivative of the given function.

33. (a) $f(x) = 4\sec(x^4)$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left(4\sec\left(x^4\right)\right)$$

By Chain rule,

$$\begin{aligned} f'(x) &= 4\sec\left(x^4\right)\tan\left(x^4\right)\frac{d}{dx}\left(x^4\right) \\ &= 4\sec\left(x^4\right)\tan\left(x^4\right)\left(4x^3\right) \\ &= 16x^3\sec\left(x^4\right)\tan\left(x^4\right) \end{aligned}$$

(b) $f(x) = 4\sec^{-1}(x^4)$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left(4\sec^{-1}\left(x^4\right)\right).$$

By Chain rule,

$$\begin{aligned} f'(x) &= 4\frac{1}{x^4\sqrt{\left(x^4\right)^2 - 1}}\frac{d}{dx}\left(x^4\right) \\ &= 4\frac{1}{x^4\sqrt{x^8 - 1}}\left(4x^3\right) \\ &= \frac{16}{x\sqrt{x^8 - 1}} \end{aligned}$$

In exercises 29–34, find the derivative of the given function.

34. (a) $f(x) = \sin^{-1}(1/x)$

$$f(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\left(\sin^{-1}\left(\frac{1}{x}\right)\right).$$

By the Chain rule,

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{x}{\sqrt{x^2-1}}\left(\frac{-1}{x^2}\right)$$

$$= -\frac{1}{x\sqrt{x^2-1}}$$

(b) $f(x) = \csc^{-1}x$

Diffrentiating with respect to x,

$$f'(x) = \frac{d}{dx}\csc^{-1}(x).$$

By the Chain rule,

$$f'(x) = -\frac{1}{x\sqrt{x^2-1}}.$$

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G12 Adv Term 1

Part 2: Writing (FRQ)

End of Term 2023-24

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Question 16

a) Use the Squeeze Theorem to find limits
b) Find limits at infinity and limits that are infinite

مدرسة العروبة للتعليم الثانوي

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Page 85, 128 & 106
Exercise 29 – 32, 37, 9 – 22, 39 - 50

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
	a) استخدام نظرية الشطيرة لإيجاد النهايات	37	128
	b) Find limits at infinity and limits that are infinite.	(9-22)	106
	b) إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية	(39-50)	

29. Use numerical and graphical evidence to conjecture the value of $\lim_{x\to 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x)$.

x^2	$x^2 \sin(1/x)$
-0.1	0.0054
-0.01	5×10^{-5}
-0.001	-8×10^{-7}
0.1	-0.005
0.01	-5×10^{-5}
0.001	8×10^{-7}

Conjecture: $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

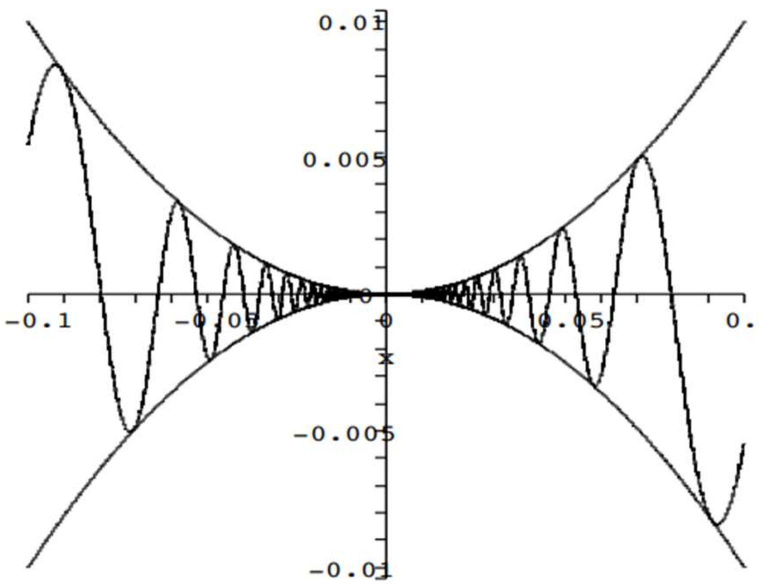
Let $f(x) = -x^2, h(x) = x^2$.

Then $f(x) \le x^2 \sin\left(\frac{1}{x}\right) \le h(x)$

$\lim_{x\to 0}(-x^2) = 0, \lim_{x\to 0}(x^2) = 0$

Therefore, by the Squeeze Theorem,

$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$



16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
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30. Why can't you use the Squeeze Theorem as in exercise 29 to prove that $\lim_{x\to 0} x^2 \sec(1/x) = 0$? Explore this limit graphically.

You cannot use the Squeeze Theorem as in exercise 29 because the secant function is not bounded between -1 and 1 like the sine function is. This is difficult to investigate graphically because of the infinitely many vertical asymptotes as x approaches 0.

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
	استخدام نظرية الشطيرة لإيجاد النهايات (a	37	128
	b) Find limits at infinity and limits that are infinite.	(9-22)	106
	إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية (b	(39-50)	

31. Use the Squeeze Theorem to prove that $\lim_{x\to 0^+}[\sqrt{x}\cos^2(1/x)] = 0$. Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x > 0$, and justify $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^+} h(x) = 0$.

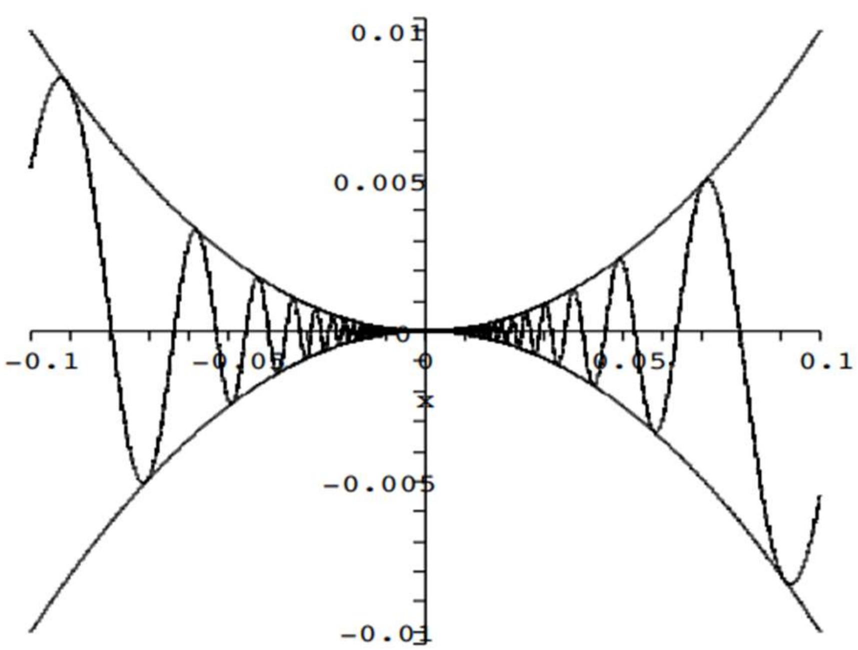
Let $f(x) = 0$, $h(x) = \sqrt{x}$. We see that

$f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$,

$\lim_{x\to 0^+} 0 = 0$, $\lim_{x\to 0^+} \sqrt{x} = 0$

Therefore, by the Squeeze Theorem,

$\lim_{x\to 0^+} \sqrt{x}\cos^2\left(\frac{1}{x}\right) = 0$.



16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
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32. Suppose that $f(x)$ is bounded: that is, there exists a constant M such that $|f(x)| \leq M$ for all x. Use the Squeeze Theorem to prove that $\lim_{x\to 0} x^2 f(x) = 0$.

Saying that $|f(x)| \leq M$ for all x is the same as saying $-M \leq f(x) \leq M$ for all x.

This implies that

$$-Mx^2 \leq x^2 f(x) \leq Mx^2.$$

Since $\pm Mx^2 \to 0$ as $x \to 0$, the Squeeze Theorem shows that $\lim_{x\to 0} x^2 f(x) = 0$.

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37. Use the Squeeze Theorem to prove that $\lim_{x \to 0} \frac{2x^3}{x^2+1} = 0$.

$$0 \le \frac{x^2}{x^2+1} < 1$$

$$\Rightarrow -2|x| \le \frac{2x^3}{x^2+1} < 2|x|$$

$$\lim_{x \to 0} -2|x| = 0; \lim_{x \to 0} 2|x| = 0$$

By the Squeeze Theorem,

$$\lim_{x \to 0} \frac{2x^3}{x^2+1} = 0.$$

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, −∞ or does not exist).

9. $\displaystyle\lim_{x\to\infty} \frac{x^2+3x-2}{3x^2+4x-1}$

$$\lim_{x\to\infty} \frac{x^2+3x-2}{3x^2+4x-1} = \lim_{x\to\infty} \frac{x^2\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}{x^2\left(3+\frac{4}{x}-\frac{1}{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(3+\frac{4}{x}-\frac{1}{x^2}\right)} = \frac{1}{3}$$

10. $\displaystyle\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}$

$$\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1} = \lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}\left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{2-1/x+1/x^2}{4-3/x-1/x^2} = \frac{1}{2}.$$

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, −∞ or does not exist).

11. $\displaystyle\lim_{x\to-\infty} \frac{-x}{\sqrt{4+x^2}}$

$$\lim_{x\to-\infty} \frac{-x}{\sqrt{4+x^2}}$$
$$= \lim_{x\to-\infty} \frac{-x}{-x\sqrt{\frac{4}{x^2}+1}}$$
$$= \lim_{x\to-\infty} \frac{1}{\sqrt{\frac{4}{x^2}+1}}$$
$$= \frac{1}{\sqrt{1}} = 1$$

12. $\displaystyle\lim_{x\to\infty} \frac{2x^2-1}{4x^3-5x-1}$

$$\lim_{x\to\infty} \frac{2x^2-1}{4x^3-5x-1} = 0.$$

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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

13. $\lim\limits_{x\to\infty} \ln\left(\dfrac{x^2+1}{x-3}\right)$

$$\lim_{x\to\infty} \ln\left(\frac{x^2+1}{x-3}\right)$$
$$= \lim_{x\to\infty}\left[\ln\left(\frac{1+\frac{1}{x^2}}{\frac{1}{x}-\frac{3}{x^2}}\right)\right]$$
$$= \lim_{x\to\infty}\left[\ln\left(\frac{1+\frac{1}{x^2}}{\frac{1}{x}-\frac{3}{x^2}}\right)\right]$$
$$= \lim_{x\to\infty}[\ln x] = \infty$$

14. $\lim\limits_{x\to 0^+} \ln(x\sin x)$

$$\lim_{x\to 0^+}[\ln(x\sin x)] = \lim_{x\to 0^+}(\ln x) = -\infty$$

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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

15. $\lim_{x \to 0^+} e^{-2/x^3}$

$$\lim_{x \to 0^+} e^{-\frac{2}{x^3}} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

16. $\lim_{x \to \infty} e^{-(x+1)/(x^2+2)}$

$$\lim_{x \to \infty} e^{\frac{-(x+1)}{(x^2+2)}} = \lim_{x \to \infty} e^{\left(\frac{-x^2\left(\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(1+\frac{2}{x^2}\right)}\right)}$$

$$= \lim_{x \to \infty} e^{\left[\frac{-\left(\frac{1}{x}+\frac{1}{x^2}\right)}{\left(1+\frac{2}{x^2}\right)}\right]} = 1$$

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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, −∞ or does not exist).

17. $\lim_{x\to\infty} \cot^{-1} x$

$$\lim_{x\to\infty} \cot^{-1} x = 0.$$

(Compare Example 5.8) We are looking for the angle that θ must approach as $\cot\theta$ goes to ∞. Look at the graph of $\cot\theta$. To define the inverse cotangent, you must pick one branch of this graph, and the standard choice is the branch immediately to the right of the y-axis. Then as $\cot\theta$ goes to ∞, the angle goes to 0.

18. $\lim_{x\to\infty} \sec^{-1} \dfrac{x^2+1}{x+1}$

$$\lim_{x\to\infty} \sec^{-1}\left(\frac{x^2+1}{x+1}\right)$$
$$= \lim_{x\to\infty} \sec^{-1}\left[\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(\frac{1}{x}+\frac{1}{x^2}\right)}\right]$$
$$= \lim_{x\to\infty} \sec^{-1}(x) = \frac{\pi}{2}$$

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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, −∞ or does not exist).

19. $\lim_{x\to 0} \sin(e^{-1/x^2})$

$$\lim_{x\to 0^+}\left(\sin\left(e^{-\frac{1}{x^2}}\right)\right) = \lim_{x\to -\infty}\left(\sin\left(e^{x}\right)\right)$$
$$= \lim_{x\to 0^+}(\sin x) = 0,$$
$$\lim_{x\to 0^-}\left(\sin\left(e^{-\frac{1}{x^2}}\right)\right)$$
$$= \lim_{x\to -\infty}\left(\sin\left(e^{x}\right)\right)$$
$$= \lim_{x\to 0^+}(\sin x) = 0$$

and hence

$$\Rightarrow \lim_{x\to 0}\left(\sin\left(e^{-\frac{1}{x^2}}\right)\right) = 0.$$

20. $\lim_{x\to \infty} \sin(\tan^{-1} x)$

$$\lim_{x\to\infty}\sin(\tan^{-1} x) = \lim_{x\to\frac{\pi}{2}}(\sin x) = 1.$$

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In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

21. $\lim\limits_{x \to \pi/2} e^{-\tan x}$

$$\lim_{x \to \frac{\pi}{2}^-} e^{-\tan x} = \lim_{x \to \infty} e^{-x} = \lim_{x \to -\infty} e^{x} = 0, \text{ but}$$

$$\lim_{x \to \frac{\pi}{2}^+} e^{-\tan x} = \lim_{x \to -\infty} e^{-x} = \lim_{x \to \infty} e^{x} = \infty,$$

so the limit does not exist.

22. $\lim\limits_{x \to 0^+} \tan^{-1}(\ln x)$

$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

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In exercises 39–48, use graphical and numerical evidence to conjecture a value for the indicated limit.

39. $\lim\limits_{x\to\infty} \dfrac{\ln(x+2)}{\ln(x^2+3x+3)}$

When x is large, the value of the fraction is very close to $\frac{1}{2}$.

40. $\lim\limits_{x\to\infty} \dfrac{\ln(2+e^{2x})}{\ln(1+e^{x})}$

When x is large, the value of the fraction is very close to 3.

41. $\lim\limits_{x\to\infty} \dfrac{x^2-4x+7}{2x^2+x\cos x}$

When x is large, the value of the fraction is very close to $\frac{1}{2}$.

42. $\lim\limits_{x\to-\infty} \dfrac{2x^3+7x^2+1}{x^3-x\sin x}$

When x is large and negative, the value of the fraction is very close to 2.

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In exercises 39–48, use graphical and numerical evidence to conjecture a value for the indicated limit.

43. $\lim_{x\to\infty} \frac{x^3+4x+5}{e^{x/2}}$

$$\lim_{x\to\infty} \frac{x^3+4x+5}{e^{x/2}} = 0.$$

44. $\lim_{x\to\infty} (e^{x/3} - x^4)$

$$\lim_{x\to\infty} (e^{x/3} - x^4) = \infty.$$

45. $\lim_{x\to 0} \frac{e^x - 1}{x}$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

46. $\lim_{x\to 0} \frac{\ln x^2}{x^2}$

$$\lim_{x\to 0} \frac{\ln(x^2)}{x^2} = -\infty.$$

47. $\lim_{x\to 0^+} x^{1/\ln x}$

$$\lim_{x\to 0^+} x^{1/(\ln x)} = e \approx 2.71828$$

48. $\lim_{x\to 0^+} x^{1/x}$

$$\lim_{x\to 0^+} x^{\frac{1}{x}} = 0$$

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
	a) استخدام نظرية الشطيرة لإيجاد النهايات	37	128
	b) Find limits at infinity and limits that are infinite.	(9-22)	106
	b) إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية	(39-50)	

In exercises 49 and 50, use graphical and numerical evidence to conjecture the value of the limit. Then, verify your conjecture by finding the limit exactly.

49. $\lim_{x\to\infty}(\sqrt{4x^2-2x+1}-2x)$ (Hint: Multiply and divide by the conjugate expression: $\sqrt{4x^2-2x+1}+2x$ and simplify.)

We multiply by

$$\frac{\sqrt{4x^2-2x+1}+2x}{\sqrt{4x^2-2x+1}+2x}$$

to get:

$$\lim_{x\to\infty}(\sqrt{4x^2-2x+1}-2x)$$

$$=\lim_{x\to\infty}\frac{-2x+1}{\sqrt{4x^2-2x+1}+2x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{-2+1/x}{\sqrt{4-2/x+1/x^2}+2}$$

$$=\frac{-2}{\sqrt{4}+2}=-\frac{1}{2}.$$

16	a) Use the Squeeze Theorem to find limits.	(29-32)	85
	a) استخدام نظرية الشطيرة لإيجاد النهايات	37	128
	b) Find limits at infinity and limits that are infinite.	(9-22)	106
	b) إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية	(39-50)	

In exercises 49 and 50, use graphical and numerical evidence to conjecture the value of the limit. Then, verify your conjecture by finding the limit exactly.

50. $\lim\limits_{x\to\infty}(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3})$ (See the hint for exercise 49.)

$$\lim_{x\to\infty}(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3})$$

If we multiply by

$$\frac{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}},$$

we get

$$\lim_{x\to\infty}\frac{(5x^2+4x+7)-(5x^2+x+3)}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3})}$$

$$=\lim_{x\to\infty}\frac{3x+4}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}$$

$$=\lim_{x\to\infty}\frac{3+\frac{4}{x}}{\sqrt{5+\frac{4}{x}+\frac{7}{x^2}}+\sqrt{5+\frac{1}{x}+\frac{3}{x^2}}}$$

$$=\frac{3}{2\sqrt{5}}=\frac{3\sqrt{5}}{10}$$

مؤسسة الإمارات للتعليم المدرسي
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Best Math

Question 17

a) Find the derivative of a function at a given point
b) Sketch the graph of a function using the graph of its derivative.

Page 145 & 151
Example 2.2
Exercise 1 – 12, 13 - 18

المدرسة الإماراتية
مدرسة العروبة للتعليم الثانوي
Justin Dsouza
Teacher

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

Example 2.2 Finding the Derivative at an Unspecified Point

Find the derivative of $f(x) = 3x^3 + 2x - 1$ at an unspecified value of x. Then, evaluate the derivative at $x = 1$, $x = 2$ and $x = 3$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^3 + 2(x+h) - 1] - (3x^3 + 2x - 1)}{h}$$

$$= \lim_{h\to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) + (2x + 2h) - 1 - 3x^3 - 2x + 1}{h}$$

Multiply out and cancel.

$$= \lim_{h\to 0} \frac{9x^2h + 9xh^2 + 3h^3 + 2h}{h}$$

Factor out common h and cancel.

$$= \lim_{h\to 0} (9x^2 + 9xh + 3h^2 + 2)$$

$$= 9x^2 + 0 + 0 + 2 = 9x^2 + 2.$$

Notice that in this case, we have derived a new *function*, $f'(x) = 9x^2 + 2$. Simply substituting in for x, we get $f'(1) = 9 + 2 = 11$
$f'(2) = 9(4) + 2 = 38$ and $f'(3) = 9(9) + 2 = 83$.

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	151
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

1. $f(x) = 3x + 1, a = 1$

Using (2.1):

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{3(1+h) + 1 - (4)}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Using (2.2):

$$\lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}$$
$$= \lim_{b \to 1} \frac{3b + 1 - (3+1)}{b - 1}$$
$$= \lim_{b \to 1} \frac{3b - 3}{b - 1}$$
$$= \lim_{b \to 1} \frac{3(b-1)}{b - 1}$$
$$= \lim_{b \to 1} 3 = 3$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	151
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In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

2. $f(x) = 3x^2 + 1, a = 1$

Using (2.1):

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2 + 1 - 4}{h} \\ &= \lim_{h\to 0} \frac{6h + 3h^2}{h} \\ &= \lim_{h\to 0} 6 + 3h = 6 \end{aligned}$$

Using (2.2):

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{(3x^2 + 1) - 4}{x - 1} \\ &= \lim_{x\to 1} \frac{3(x-1)(x+1)}{x - 1} \\ &= \lim_{x\to 1} 3(x+1) = 6 \end{aligned}$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
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	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

3. $f(x) = \sqrt{3x+1}$, $a = 1$

Using (2.1): Since

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{3(1+h)+1}-2}{h} = \frac{\sqrt{4+3h}-2}{h}\cdot\frac{\sqrt{4+3h}+2}{\sqrt{4+3h}+2}$$
$$= \frac{4+3h-4}{h(\sqrt{4+3h}+2)} = \frac{3h}{h(\sqrt{4+3h}+2)}$$
$$= \frac{3}{\sqrt{4+3h}+2} = \frac{3}{\sqrt{4+3h}+2},$$

we have

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{3}{\sqrt{4+3h}+2} = \frac{3}{\sqrt{4+3(0)}+2} = \frac{3}{4}$$

Using (2.2): Since

$$\frac{f(b)-f(1)}{b-1} = \frac{\sqrt{3b+1}-2}{b-1} = \frac{(\sqrt{3b+1}-2)(\sqrt{3b+1}+2)}{(b-1)(\sqrt{3b+1}+2)}$$
$$= \frac{(3b+1)-4}{(b-1)\sqrt{3b+1}+2} = \frac{3(b-1)}{(b-1)\sqrt{3b+1}+2} = \frac{3}{\sqrt{3b+1}+2},$$

we have

$$f'(1) = \lim_{b\to 1}\frac{f(b)-f(1)}{b-1} = \lim_{b\to 1}\frac{3}{\sqrt{3b+1}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
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In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

4. $f(x) = \dfrac{3}{x+1}$, $a = 2$

Using (2.1):

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{(2+h)+1} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3+h} - \frac{3+h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{3+h} = -\frac{1}{3}$$

Using (2.2):

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x\to 2} \frac{\frac{3}{x+1} - 1}{x-2}$$

$$= \lim_{x\to 2} \frac{\frac{3}{x+1} - \frac{x+1}{x+1}}{x-2}$$

$$= \lim_{x\to 2} \frac{\frac{-(x-2)}{x+1}}{x-2}$$

$$= \lim_{x\to 2} \frac{-1}{x+1} = -\frac{1}{3}$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 5–12, compute the derivative function f' using (2.1) or (2.2).

5. $f(x) = 3x^2 + 1$

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 + 1 - (3(x)^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 1 - (3x^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h\to 0} (6x + 3h) = 6x
\end{aligned}$$

6. $f(x) = x^2 - 2x + 1$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) + 1 - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h - 2)}{h} = 2x - 2
\end{aligned}$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 5–12, compute the derivative function f' using (2.1) or (2.2).

7. $f(x) = x^3 + 2x - 1$

$$\lim_{b\to x} \frac{f(b)-f(x)}{b-x}$$
$$= \lim_{b\to x} \frac{b^3+2b-1-\left(x^3+2x-1\right)}{b-x}$$
$$= \lim_{b\to x} \frac{(b-x)\left(b^2+bx+x^2+2\right)}{b-x}$$
$$= \lim_{b\to x} b^2+bx+x^2+2$$
$$= 3x^2+2$$

8. $f(x) = x^4 - 2x^2 + 1$

$$f'(x) =$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^4-2(x+h)^2+1-f(x)}{h}$$
$$= \lim_{h\to 0} \left[4x^3+6x^2h+4xh^2+h^3-4x-2h\right]$$
$$= 4x^3-4x$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
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	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
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In exercises 5–12, compute the derivative function f' using (2.1) or (2.2).

9. $f(x) = \dfrac{3}{x+1}$

$$\lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$
$$= \lim_{b \to x} \frac{\frac{3}{b+1} - \frac{3}{x+1}}{b - x}$$
$$= \lim_{b \to x} \frac{\frac{3(x+1) - 3(b+1)}{(b+1)(x+1)}}{b - x}$$
$$= \lim_{b \to x} \frac{-3(b - x)}{(b+1)(x+1)(b - x)}$$
$$= \lim_{b \to x} \frac{-3}{(b+1)(x+1)}$$
$$= \frac{-3}{(x+1)^2}$$

10. $f(x) = \dfrac{2}{2x - 1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2}{2(x+h)-1} - \frac{2}{2x-1}}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2(2x-1) - 2(2x+2h-1)}{(2x+2h-1)(2x-1)}}{h}$$
$$= \lim_{h \to 0} \frac{\frac{-4h}{(2x+2h-1)(2x-1)}}{h}$$
$$= \lim_{h \to 0} \frac{-4}{(2x + 2h - 1)(2x - 1)}$$
$$= \frac{-4}{(2x - 1)^2}$$

17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
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In exercises 5–12, compute the derivative function f' using (2.1) or (2.2).

11. $f(t) = \sqrt{3t+1}$

$$f'(t) = \lim_{b\to t} \frac{f(b) - f(t)}{(b-t)}$$

$$= \lim_{b\to t} \frac{\sqrt{3b+1} - \sqrt{3t+1}}{(b-t)}$$

Multiplying by $\dfrac{\sqrt{3b+1}+\sqrt{3t+1}}{\sqrt{3b+1}+\sqrt{3t+1}}$ gives

$$f'(t) = \lim_{b\to t} \frac{(3b+1) - (3t+1)}{(b-t)\left(\sqrt{3b+1}+\sqrt{3t+1}\right)}$$

$$= \lim_{b\to t} \frac{3(b-t)}{(b-t)\left(\sqrt{3b+1}+\sqrt{3t+1}\right)}$$

$$= \lim_{b\to t} \frac{3}{\sqrt{3b+1}+\sqrt{3t+1}}$$

$$= \frac{3}{2\sqrt{3t+1}}$$

12. $f(t) = \sqrt{2t+4}$

$$f'(t) = \lim_{b\to t} \frac{f(b) - f(t)}{(b-t)}$$

$$= \lim_{b\to t} \frac{\sqrt{2b+4} - \sqrt{2t+4}}{(b-t)}$$

Multiplying by $\dfrac{\sqrt{2b+4}+\sqrt{2t+4}}{\sqrt{2b+4}+\sqrt{2t+4}}$ gives

$$f'(t) = \lim_{b\to t} \frac{(2b+4) - (2t+4)}{(b-t)\left(\sqrt{2b+4}+\sqrt{2t+4}\right)}$$

$$= \lim_{b\to t} \frac{2(b-t)}{(b-t)\left(\sqrt{2b+4}+\sqrt{2t+4}\right)}$$

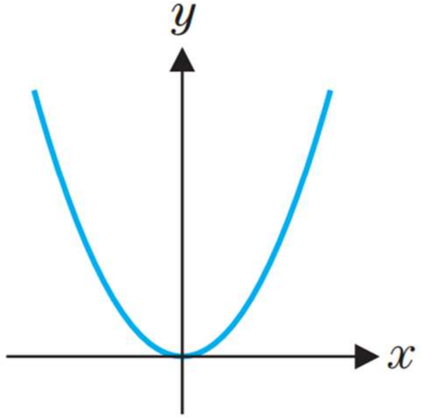
$$= \lim_{b\to t} \frac{2}{\sqrt{2b+4}+\sqrt{2t+4}}$$

$$= \frac{2}{2\sqrt{2t+4}} = \frac{1}{\sqrt{2t+4}}$$

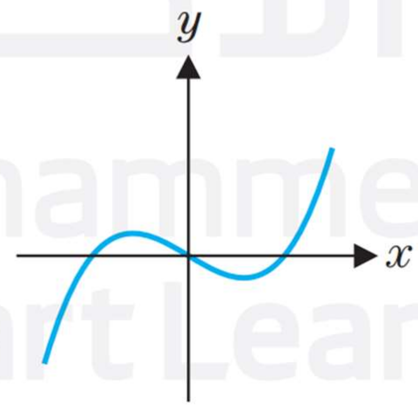
17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	151
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 13–16, use the graph of f to sketch a graph of f'.

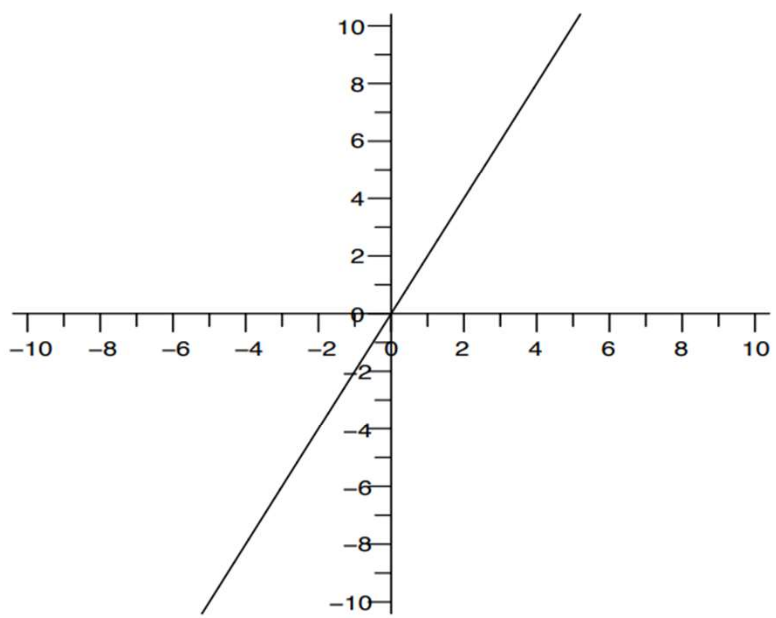
13. (a)



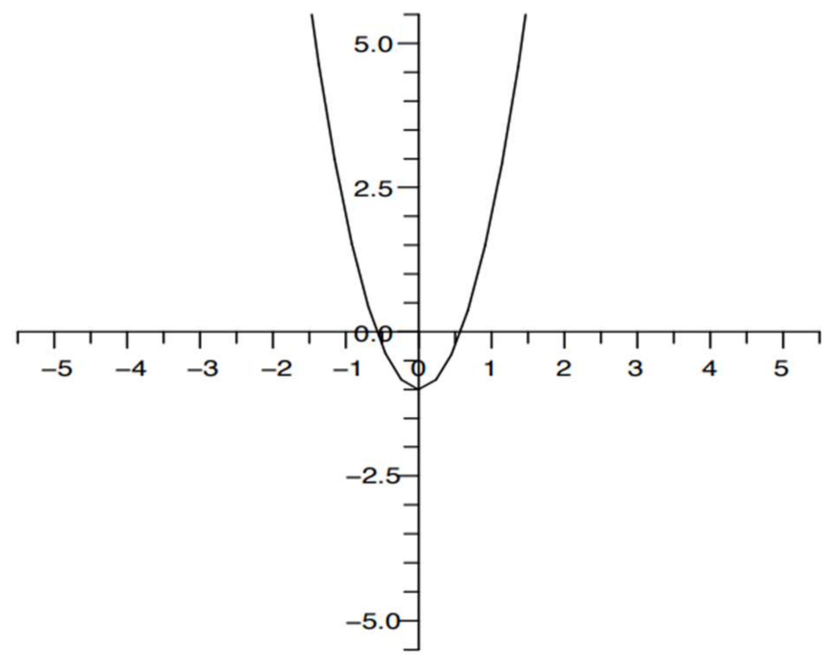
(b)



(a) The derivative should look like:



(b) The derivative should look like:



17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
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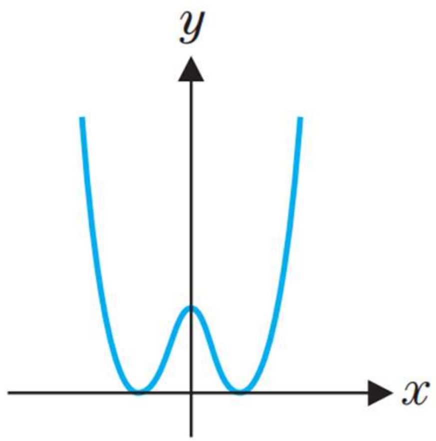
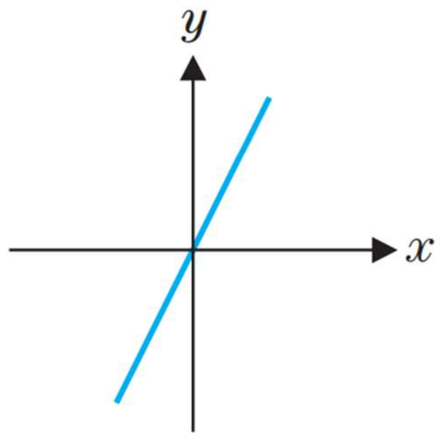
In exercises 13–16, use the graph of f to sketch a graph of f'.

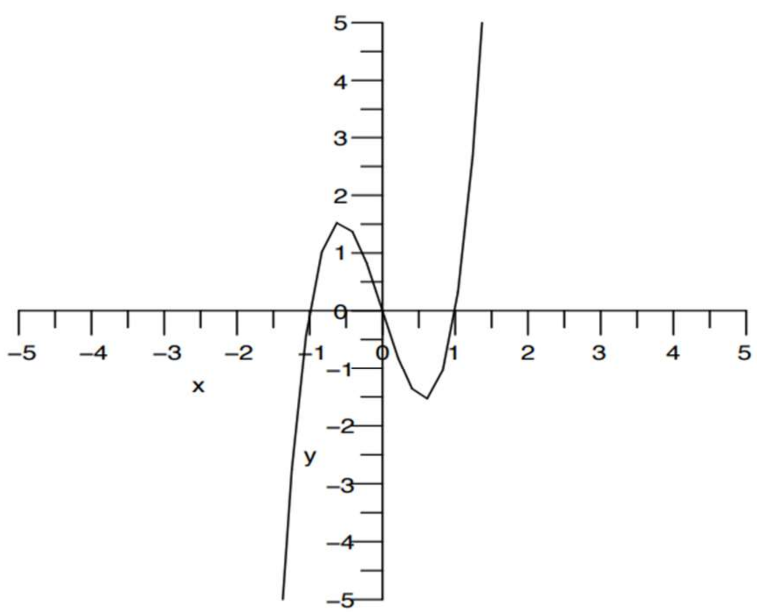
14. (a)



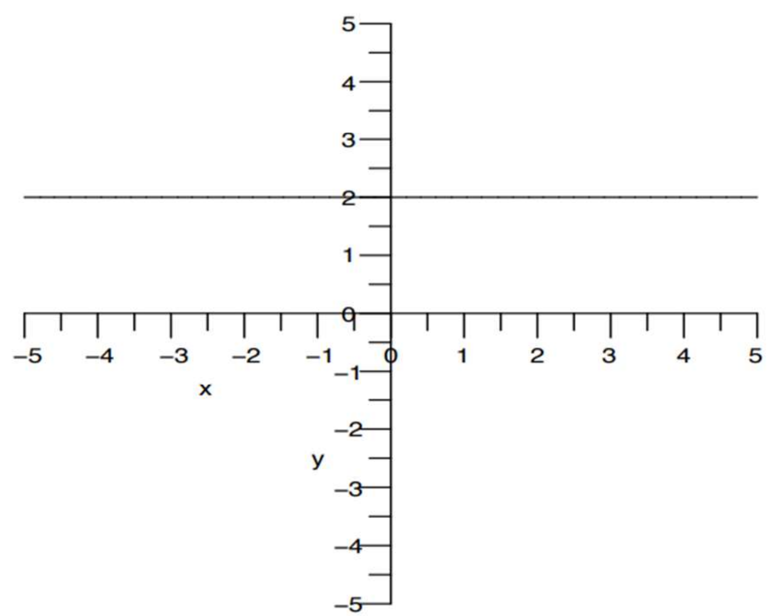
(b)



(a) The derivative should look like:



(b) The derivative should look like:



17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
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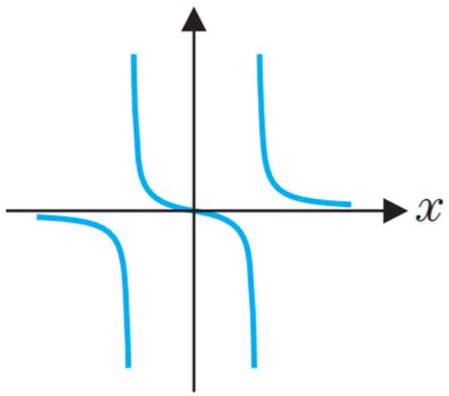
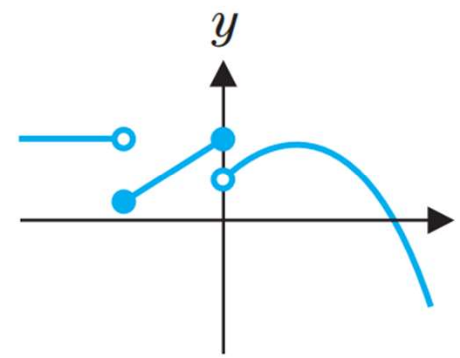
In exercises 13–16, use the graph of f to sketch a graph of f'.

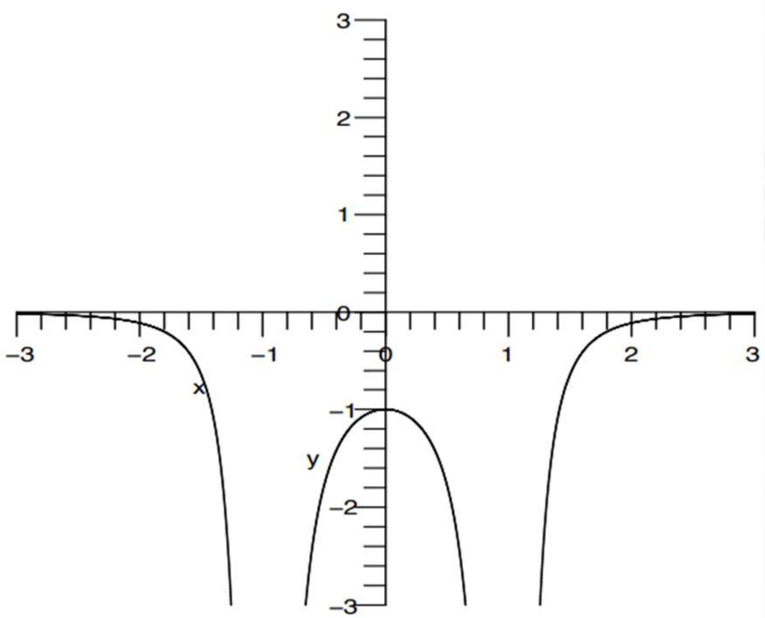
15. (a)



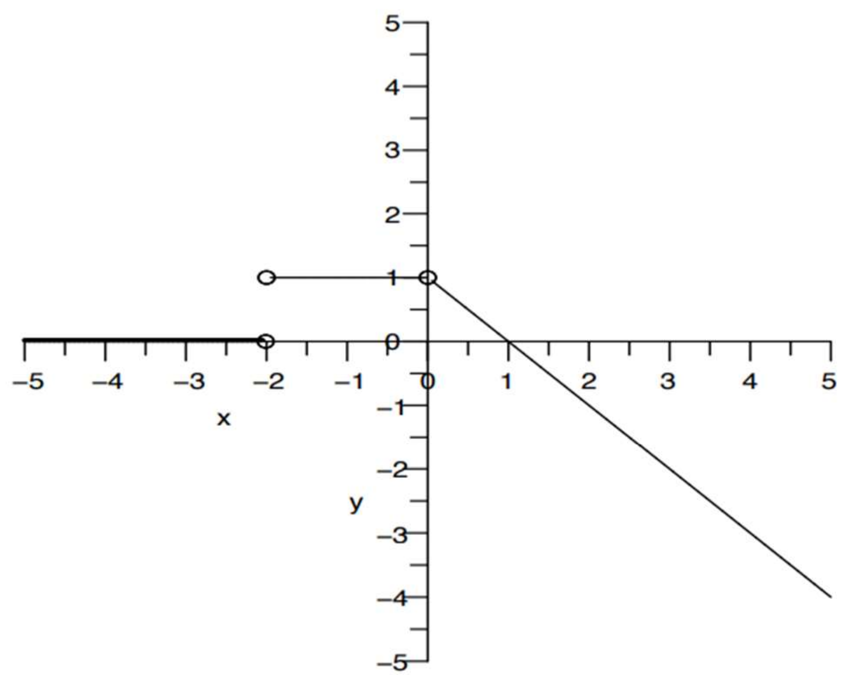
(b)



(a) The derivative should look like:



(b) The derivative should look like:



17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
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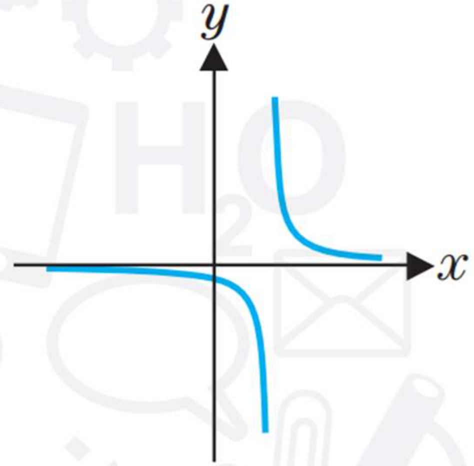
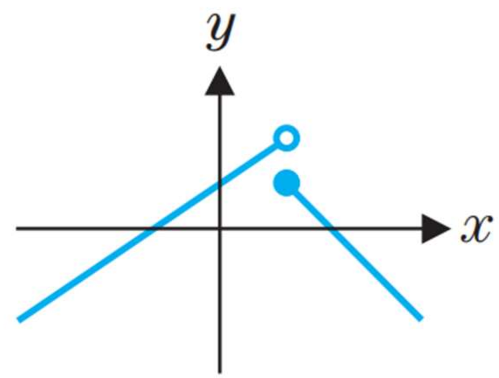
In exercises 13–16, use the graph of f to sketch a graph of f'.

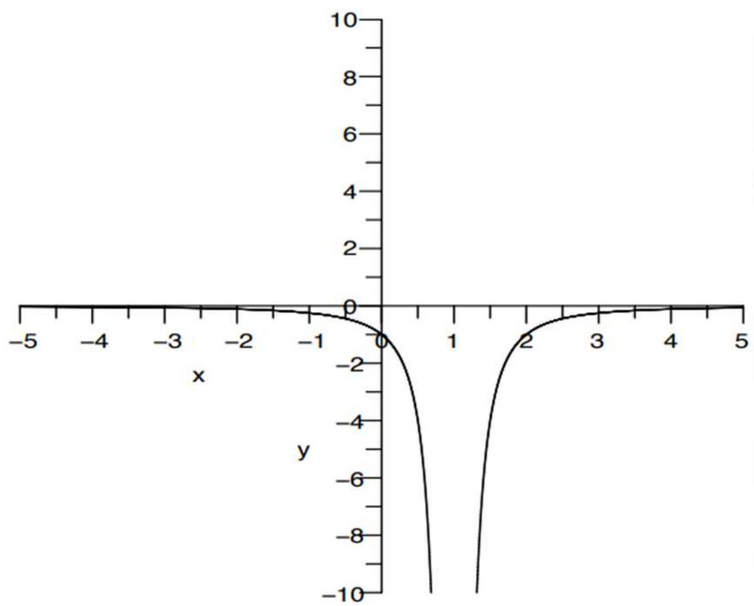
16. (a)



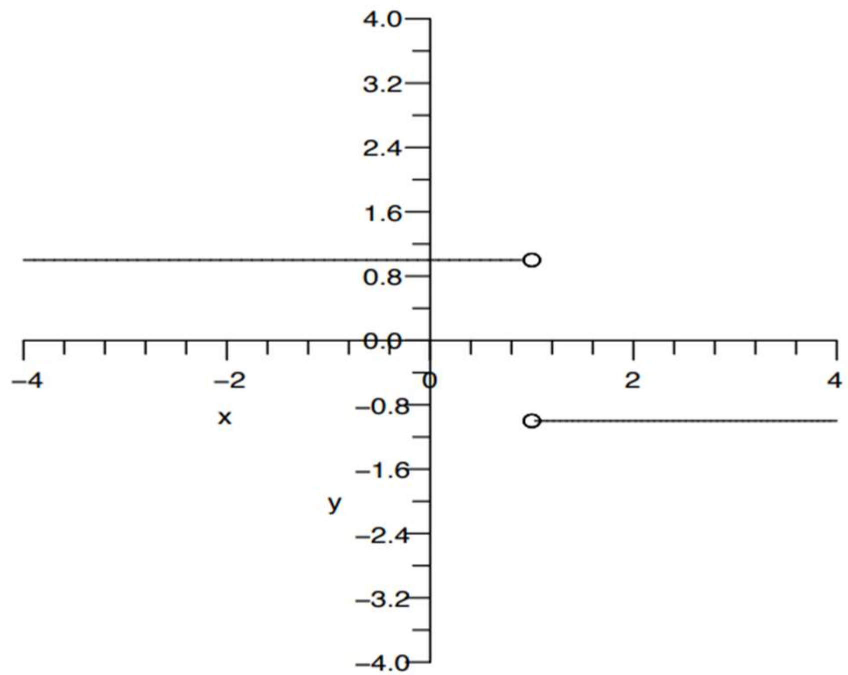
(b)



(a) The derivative should look like:



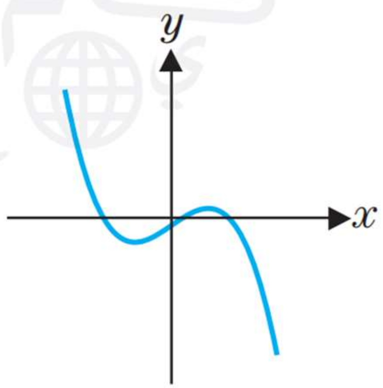
(b) The derivative should look like:



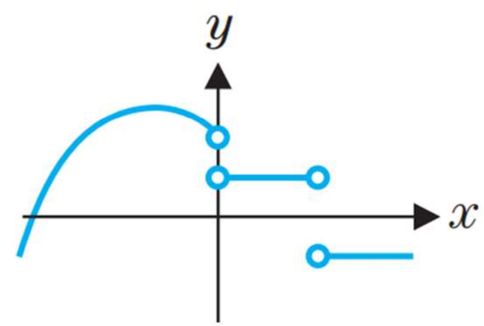
17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 17 and 18, use the given graph of f' to sketch a plausible graph of a continuous function f.

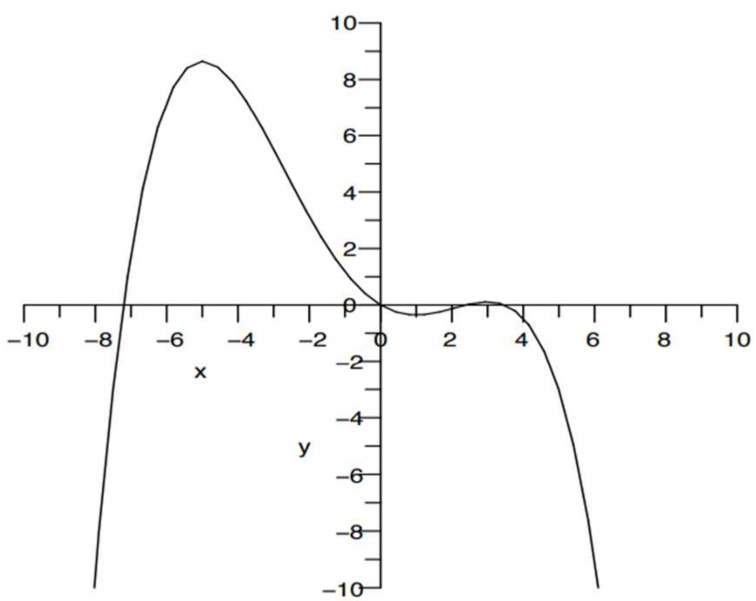
17. (a)



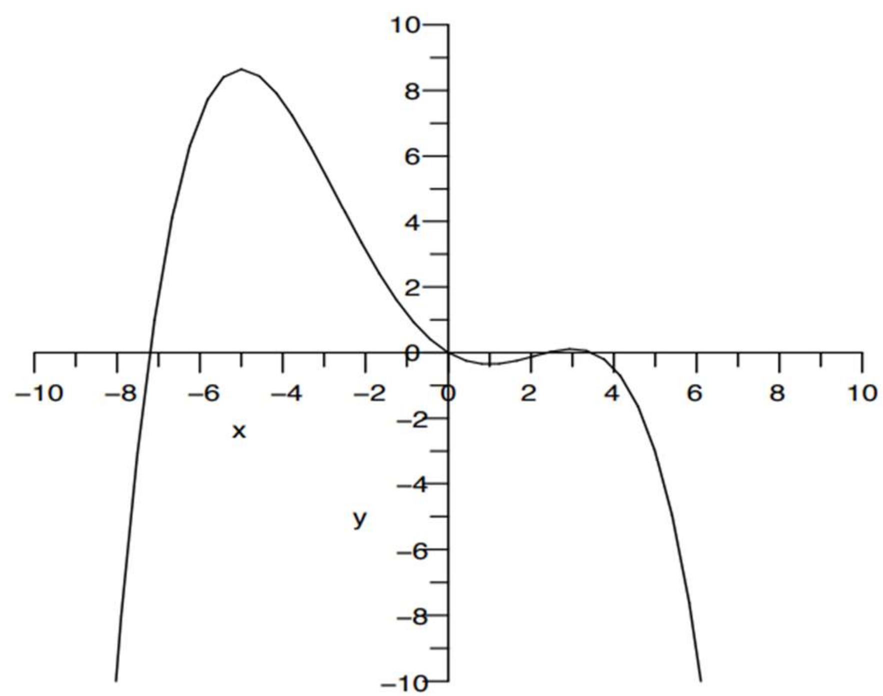
(b)



(a) The function should look like:



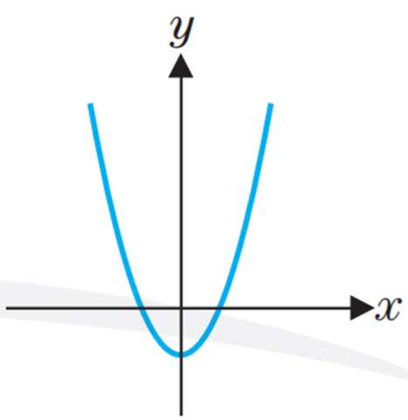
(b) The derivative should look like:



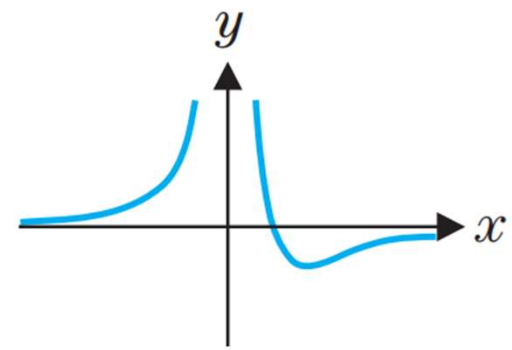
17	a) Find the derivative of a function at a given point.	Example2.2	145
	a) إيجاد المشتقة لدالة عند نقطة ما	(1-12)	151
	b) Sketch the graph of a function using the graph of its derivative.	(13-18)	
	b) رسم منحنى الدالة اعتمادا على التمثيل البياني لمشتقتها		

In exercises 17 and 18, use the given graph of f' to sketch a plausible graph of a continuous function f.

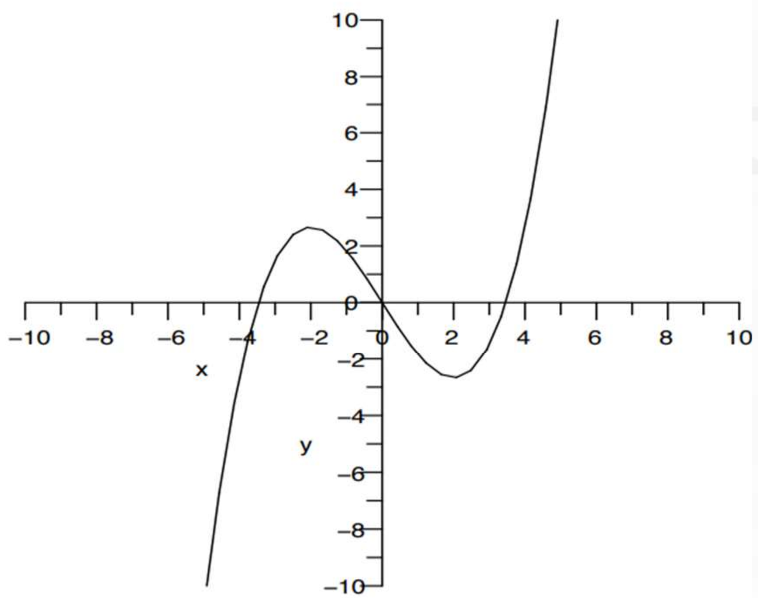
18. (a)



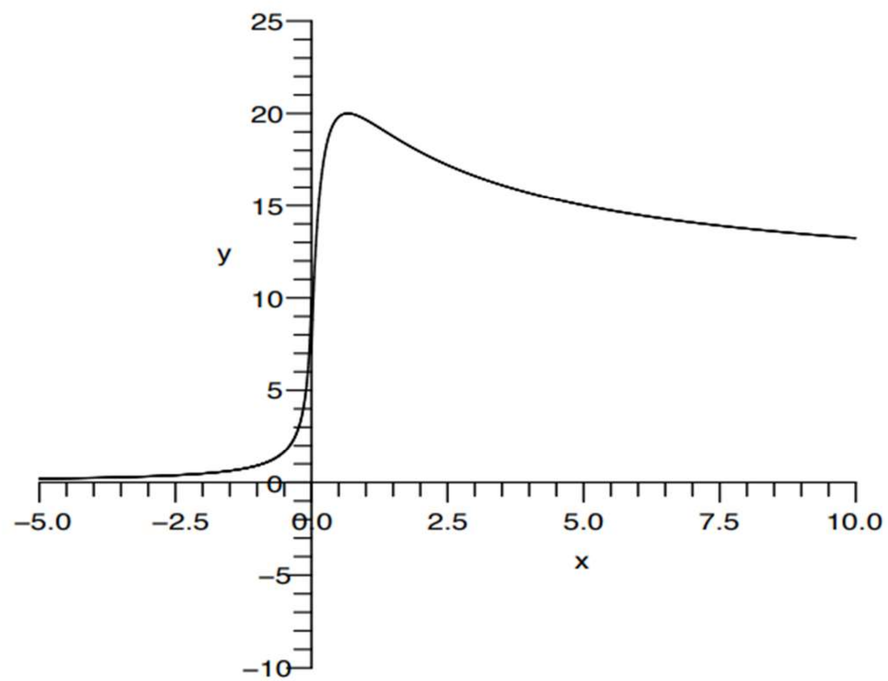
(b)



18. (a) The function should look like:



(b) The derivative should look like:



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Question 18

Solve real-life problems using derivatives of exponential and logarithmic functions

Page 192 & 194
Example 7.5
Exercise 37 – 38

المدرسة الإماراتية
مدرسة العروبة للتعليم الثانوي
Justin Dsouza
Teacher

18	Solve real-life problems using derivatives of exponential and logarithmic functions.	Example 7.5	192
	حل مسائل حياتية باستخدام مشتقات الدوال الأسية واللوغاريتمية الطبيعية	(37,38)	194

Example 7.5 Analyzing the Concentration of a Chemical

The concentration c of a certain chemical after t seconds of an autocatalytic reaction is given by $c(t) = \dfrac{10}{9e^{-20t} + 1}$. Show that $c'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 10.

Solution:

$$c(t) = 10(9e^{-20t} + 1)^{-1}$$

$$c'(t) = -10(9e^{-20t} + 1)^{-2}\frac{d}{dt}(9e^{-20t} + 1)$$

$$= -10(9e^{-20t} + 1)^{-2}(-180e^{-20t})$$

$$= 1800e^{-20t}(9e^{-20t} + 1)^{-2}$$

$$= \frac{1800e^{-20t}}{(9e^{-20t} + 1)^2} > 0.$$

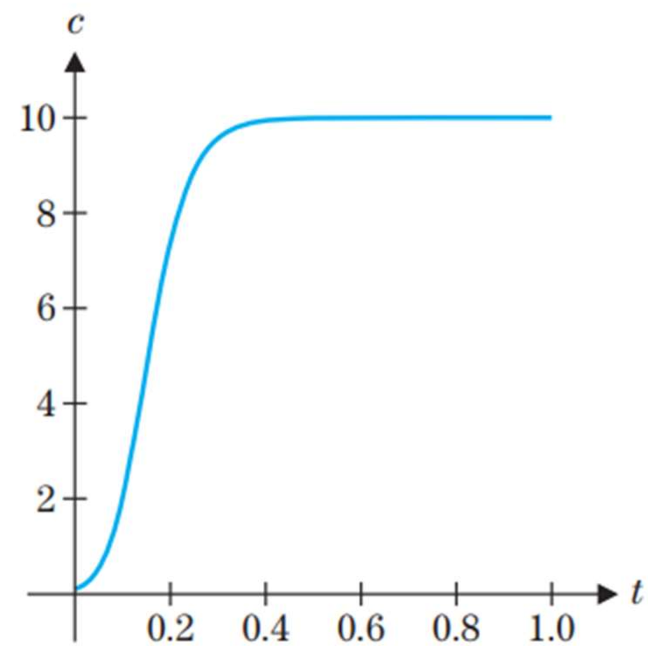

FIGURE 3.38
Chemical concentration

Since all of the tangent lines have positive slope, the graph of $y = c(t)$ rises from left to right, as shown in Figure 3.38.

Since the concentration increases for all time, the concentration is always less than the limiting value $\lim_{t\to\infty} c(t)$, which is easily computed to be

$$\lim_{t\to\infty} \frac{10}{9e^{-20t} + 1} = \frac{10}{0 + 1} = 10.$$

18	Solve real-life problems using derivatives of exponential and logarithmic functions.	Example 7.5	192
	حل مسائل حياتية باستخدام مشتقات الدوال الأسية واللوغاريتمية الطبيعية	(37,38)	194

37. The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $c(t) = \dfrac{6}{2e^{-8t}+1}$. Show that $c'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 6.

$$c(t) = \frac{6}{2e^{-8t}+1} = 6\left(2e^{-8t}+1\right)^{-1}$$

$$c'(t) = -6\left(2e^{-8t}+1\right)^{-2}.\left(-16e^{-8t}\right)$$

$$= \frac{96e^{-8t}}{\left(2e^{-8t}+1\right)^{2}}$$

Since $e^{-8t} > 0$ for any t both numerator and denominator are positive,so that $c'(t) > 0$. Then, since $c(t)$ is an increasing function with a limiting value of 6 (as t goes to infinity) the concentration never exceeds (indeed, never reaches) the value of 6.

18	Solve real-life problems using derivatives of exponential and logarithmic functions.	Example 7.5	192
	حل مسائل حياتية باستخدام مشتقات الدوال الأسية واللوغاريتمية الطبيعية	(37,38)	194

38. The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $c(t) = \dfrac{10}{9e^{-10t} + 2}$. Show that $c'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 5.

$$c'(t) = -10\left(9e^{-10t} + 2\right)^{-2}\left(-90e^{-10t}\right)$$

$$= \frac{900e^{-10t}}{\left(9e^{-10t} + 2\right)^2}$$

Since $e^{-10t} > 0$ for all t, $c'(t) > 0$ for all t, and $c(t)$ is increasing for all t. This forces, $c(t) < \lim_{t \to \infty} c(t) = 5$

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Question 19

Find derivatives implicitly

Page 198, 204 & 222

Example 8.2

Exercise 1 – 16, 13 - 14

المدرسة الإماراتية

مدرسة العروبة للتعليم الثانوي

Justin Dsouza

Teacher

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

Example 8.2 Finding a Tangent Line by Implicit Differentiation

Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$. Then, find an equation of the tangent line at the point $(2, -2)$.

Solution:

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^2y^2 - 2x) = \frac{d}{dx}(4 - 4y).$$

Since the first term is the product of x^2 and y^2, we must use the product rule. We get

$$2xy^2 + x^2(2y)y'(x) - 2 = 0 - 4y'(x).$$

Grouping the terms with $y'(x)$ on one side, we get

$$(2x^2y + 4)y'(x) = 2 - 2xy^2$$

$$y'(x) = \frac{2 - 2xy^2}{2x^2y + 4}.$$

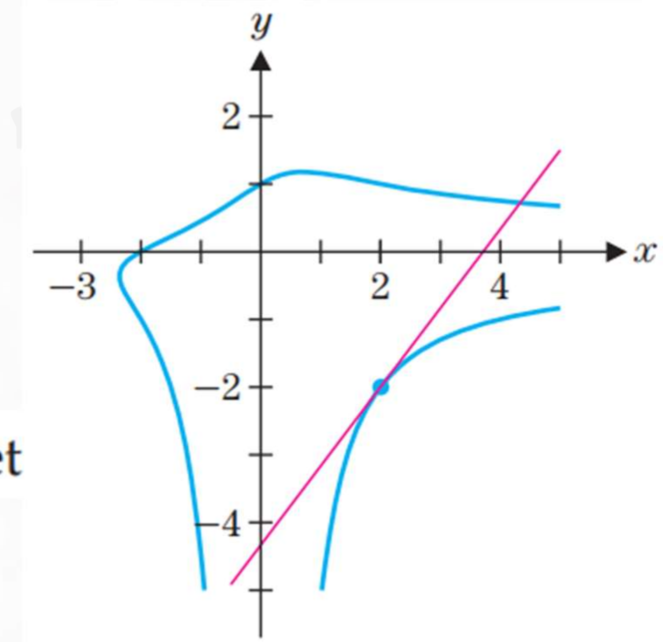

FIGURE 3.41
Tangent line at $(2, -2)$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
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Example 8.2 Finding a Tangent Line by Implicit Differentiation

Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$. Then, find an equation of the tangent line at the point $(2, -2)$.

Solution (Continued):

Substituting $x = 2$ and $y = -2$, we get the slope of the tangent line,

$$y'(2) = \frac{2 - 16}{-16 + 4} = \frac{7}{6}.$$

Finally, an equation of the tangent line is given by

$$y + 2 = \frac{7}{6}(x - 2).$$

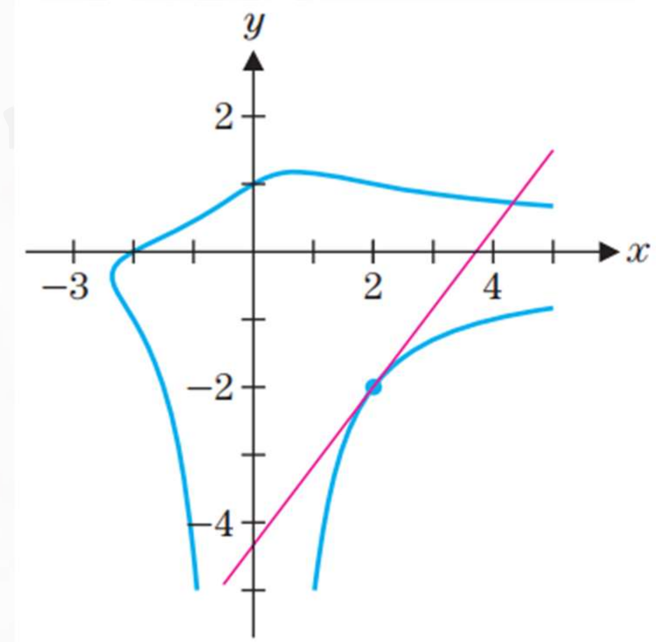

FIGURE 3.41
Tangent line at $(2, -2)$

We have plotted the curve and the tangent line at $(2, -2)$ in Figure 3.41 using the implicit plot mode of our computer algebra system.

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
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In exercises 1–4, compute the slope of the tangent line at the given point both explicitly (first solve for y as a function of x) and implicitly.

1. $x^2 + 4y^2 = 8$ at $(2, 1)$

Explicitly:

$$4y^2 = 8 - x^2$$

$$y^2 = \frac{8 - x^2}{4}$$

$$y = \pm\frac{\sqrt{8 - x^2}}{2} \text{ (choose plus to fit(2,1))}$$

$$\text{For } y = \frac{\sqrt{8 - x^2}}{2},$$

$$y' = \frac{1}{2}\frac{(-2x)}{2\sqrt{8 - x^2}} = \frac{-x}{2\sqrt{8 - x^2}},$$

$$y'(2) = \frac{-1}{2}.$$

Implicitly:

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(8)$$

$$2x + 8y \cdot y' = 0$$

$$y' = -\frac{2x}{8y} = -\frac{x}{4y}$$

$$\text{At } (2,\ 1):\ y' = -\frac{2}{4} = -\frac{1}{2}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 1–4, compute the slope of the tangent line at the given point both explicitly (first solve for y as a function of x) and implicitly.

2. $x^3y - 4\sqrt{x} = x^2y$ at $(2, \sqrt{2})$

Explicitly:

$$y = \frac{4\sqrt{x}}{x^3 - x^2}$$

$$y' = \frac{\left(x^3 - x^2\right)\frac{2}{\sqrt{x}} - 4\sqrt{x}\left(3x^2 - 2x\right)}{\left(x^3 - x^2\right)^2}$$

Implicitly differentiating:

$$3x^2y + x^3y' - \frac{2}{\sqrt{x}} = 2xy + x^2y',$$

And we solve for y' to get

$$y' = \frac{2xy + \frac{2}{\sqrt{x}} - 3x^2y}{x^3 - x^2}.$$

Substitute $x = 2$ into the first expression, and $(x, y) = \left(2, \sqrt{2}\right)$, into the second to get $y' = -\dfrac{7\sqrt{2}}{4}$.

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 1–4, compute the slope of the tangent line at the given point both explicitly (first solve for y as a function of x) and implicitly.

3. $y - 3x^2y = \cos x$ at $(0, 1)$

Explicitly:

$$y(1-3x^2) = \cos x$$

$$y = \frac{\cos x}{1-3x^2}$$

$$y'(x) = \frac{(1-3x^2)(-\sin x) - \cos x(-6x)}{(1-3x^2)^2}$$

$$= \frac{-\sin x + 3x^2 \sin x + 6x\cos x}{(1-3x^2)^2}$$

$$y'(0) = 0.$$

Implicitly:

$$\frac{d}{dx}(y - 3x^2y) = \frac{d}{dx}(\cos x)$$

$$y' - 3x^2y' - 6xy = -\sin x$$

$$y'(1-3x^2) = 6xy - \sin x$$

$$y' = \frac{6xy - \sin x}{1-3x^2}$$

At $(0,1)$: $y' = 0$(again)

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 1–4, compute the slope of the tangent line at the given point both explicitly (first solve for y as a function of x) and implicitly.

4. $y^2 + 2xy + 4 = 0$ at $(-2, 2)$

Explicitly:

$y = -x \pm \sqrt{x^2 - 4}$

At the point $(-2, 2)$, the sign is irrelevant, so we choose

$y = -x + \sqrt{x^2 - 4}$

$$y' = -1 + \frac{2x}{2\sqrt{x^2 - 4}} = -1 + \frac{x}{\sqrt{x^2 - 4}}$$

Implicitly differentiating:

$y' + 2y + 2xy' = 0,$

and we solve for y' :

$$y' = \frac{-2y}{2x + 2y}$$

Substitute $x = -2$ in the first expression and $(x, y) = (-2, 2)$ in to the second expression to see that y' is undefined. There is a vertical tangent at this point.

19	**Find derivatives implicitly.**	**Example 8.2**	**198**
	إيجاد المشتقات للعلاقات الضمنية	**(1-16)**	**204**
		(13,14)	**222**

In exercises 5–16, find the derivative $y'(x)$ implicitly.

5. $x^2y^2 + 3y = 4x$

$$\frac{d}{dx}(x^2y^2 + 3y) = \frac{d}{dx}(4x)$$

$$2xy^2 + x^2 2yy' + 3y' = 4$$

$$y'(2x^2y + 3) = 4 - 2xy^2$$

$$y' = \frac{4 - 2xy^2}{2x^2y + 3}$$

6. $3xy^3 - 4x = 10y^2$

$$3y^3 + 3x(3y^2)y' - 4 = 20yy'$$

$$(9xy^2 - 20y)y' = 4 - 3y^3$$

$$y' = \frac{3y^3 - 4}{20y - 9xy^2}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 5–16, find the derivative $y'(x)$ implicitly.

7. $\sqrt{xy} - 4y^2 = 12$

$$\frac{d}{dx}(\sqrt{xy} - 4y^2) = \frac{d}{dx}(12)$$
$$\frac{1}{2\sqrt{xy}} \cdot \frac{d}{dx}(xy) - 8y \cdot y' = 0$$
$$\frac{1}{2\sqrt{xy}}.(xy' + y) - 8y \cdot y' = 0$$
$$(xy' + y) - 16y \cdot y'\sqrt{xy} = 0$$
$$y'(x - 16y\sqrt{xy}) = -y$$
$$y' = \frac{-y}{(x - 16y\sqrt{xy})} = \frac{y}{16y\sqrt{xy} - x}$$

8. $\sin xy = x^2 - 3$

$$\cos(xy)(y + xy') = 2x$$
$$y' = \frac{2x - y\cos(xy)}{x\cos(xy)}$$

9. $\frac{x+3}{y} = 4x + y^2$

$$x + 3 = 4xy + y^3$$
$$1 = \frac{d}{dx}(4xy + y^3) = 4(xy' + y) + 3y^2y'$$
$$1 - 4y = (4x + 3y^2)y'$$
$$y' = \frac{1 - 4y}{3y^2 + 4x}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 5–16, find the derivative $y'(x)$ implicitly.

10. $3x + y^3 - \dfrac{4y}{x+2} = 10x^2$

Diffrentiating with respect to x,

$$\frac{d}{dx}\left(3x + y^3 - \frac{4y}{x+2}\right) = \frac{d}{dx}\left(10x^2\right)$$

By the Chain rule and Product rule,

$$3 + 3y^2y' - \left[\frac{(x+2)\,4y' - 4y}{(x+2)^2}\right] = 20x$$

$$3(x+2)^2 + 3y^2y'(x+2)^2$$
$$-4y'\,(x+2) + 4y = 20x(x+2)^2$$
$$3y^2y'(x+2)^2 - 4y'\,(x+2)$$
$$= 20x(x+2)^2 - 3(x+2)^2 - 4y$$
$$y'\,(x+2)\left[3y^2\,(x+2) - 4\right]$$
$$= (x+2)^2\,(20x-3) - 4y$$
$$y' = \frac{(x+2)^2\,(20x-3) - 4y}{(x+2)\left[3y^2\,(x+2) - 4\right]}$$

11. $e^{x^2y} - e^y = x$

$$\frac{d}{dx}(e^{x^2y} - e^{y)}) = \frac{d}{dx}(x)$$
$$e^{x^2y}\frac{d}{dx}(e^{x^2y}) - e^yy' = 1$$
$$e^{x^2y}(2xy + x^2y') - e^yy' = 1$$
$$y'(x^2e^{x^2y} - e^y) = 1 - 2xye^{x^2y}$$
$$y' = \frac{1 - 2xye^{x^2}}{(x^2e^{x^2y} - e^y)}$$

12. $xe^y - 3y\sin x = 1$

$$e^y + xe^yy' - 3y'\sin x - 3y\cos x = 0$$
$$y' = \frac{3y\cos x - e^y}{xe^y - 3\sin x}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 5–16, find the derivative $y'(x)$ implicitly.

13. $y^2\sqrt{x+y}-4x^2=y$

Diffrentiating with respect to x,

$$\frac{d}{dx}\left(y^2\sqrt{x+y}-4x^2\right)=\frac{d}{dx}(y)$$

By the Chain rule and Product rule,

$$\frac{d}{dx}\left(y^2\sqrt{x+y}\right)-4\frac{d}{dx}\left(x^2\right)=\frac{d}{dx}(y)$$

$$\left[y^2\left(\frac{1}{2\sqrt{x+y}}\right)(1+y')\right]+2yy'\sqrt{x+y}-8x=y'$$

$$y^2+y^2y'+4yy'(x+y)-16x\sqrt{x+y}=2y'\sqrt{x+y}$$

$$y^2y'+4yy'(x+y)-2y'\sqrt{x+y}=16x\sqrt{x+y}-y^2$$

$$y'\left[y^2+4y(x+y)-2\sqrt{x+y}\right]=16x\sqrt{x+y}-y^2$$

$$y'=\frac{16x\sqrt{x+y}-y^2}{y^2+4y(x+y)-2\sqrt{x+y}}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 5–16, find the derivative $y'(x)$ implicitly.

14. $x\cos(x+y) - y^2 = 8$

Diffrentiating with respect to x,

$$\frac{d}{dx}\left(x\cos(x+y) - y^2\right) = \frac{d}{dx}(8)$$

By the Chain rule and Product rule,

$$\frac{d}{dx}\left(x\cos(x+y)\right) - \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(8)$$

$$\cos(x+y) - x\sin(x+y)(1+y') - 2yy' = 0$$

$$\cos(x+y) - x\sin(x+y) - x\sin(x+y)\,y'$$

$$-2yy' = 0$$

$$y'\left(-x\sin(x+y) - 2y\right)$$

$$= x\sin(x+y) - \cos(x+y)$$

$$y' = \frac{x\sin(x+y) - \cos(x+y)}{-x\sin(x+y) - 2y}$$

$$y' = \frac{\cos(x+y) - x\sin(x+y)}{x\sin(x+y) + 2y}$$

15. $e^{4y} - \ln(y^2+3) = 2x$

$$e^{4y} - \ln\left(y^2+3\right) = 2x$$

Diffrentiating with respect to x,

$$\frac{d}{dx}\left(e^{4y} - \ln\left(y^2+3\right)\right) = \frac{d}{dx}(2x)$$

By the Chain rule and Product rule,

$$\frac{d}{dx}\left(e^{4y}\right) - \frac{d}{dx}\left(\ln\left(y^2+3\right)\right) = \frac{d}{dx}(2x)$$

$$e^{4y}(4y') - \frac{2yy'}{y^2+3} = 2$$

$$4e^{4y}\left(y^2+3\right)y' - 2yy' = 2\left(y^2+3\right)$$

$$y'\left(4e^{4y}\left(y^2+3\right) - 2y\right) = 2\left(y^2+3\right)$$

$$y' = \frac{2\left(y^2+3\right)}{4e^{4y}\left(y^2+3\right) - 2y}$$

19	Find derivatives implicitly.	Example 8.2	198
	إيجاد المشتقات للعلاقات الضمنية	(1-16)	204
		(13,14)	222

In exercises 5–16, find the derivative $y'(x)$ implicitly.

16. $e^{x^2}y - 3\sqrt{y^2+2} = x^2 + 1$

Diffrentiating with respect to x,

$$\frac{d}{dx}\left(e^{x^2}y - 3\sqrt{y^2+2}\right) = \frac{d}{dx}\left(x^2+1\right)$$

By the Chain rule and Produt rule,

$$\frac{d}{dx}\left(e^{x^2}y\right) - 3\frac{d}{dx}\left(\sqrt{y^2+2}\right) = 2x$$

$$e^{x^2}(2x)\,y + e^{x^2}y' - 3\cdot\frac{2yy'}{2\sqrt{y^2+2}} = 2x$$

$$2xye^{x^2} + e^{x^2}y' - \frac{3yy'}{\sqrt{y^2+2}} = 2x$$

$$2xye^{x^2}\sqrt{y^2+2} + e^{x^2}y'\sqrt{y^2+2} - 3yy'$$

$$= 2x\sqrt{y^2+2}$$

$$y'\left(e^{x^2}\sqrt{y^2+2} - 3y\right) = 2x\sqrt{y^2+2}$$

$$-2xye^{x^2}\sqrt{y^2+2}$$

$$y' = \frac{2x\sqrt{y^2+2}\left(1 - ye^{x^2}\right)}{e^{x^2}\sqrt{y^2+2} - 3y}$$

19	Find derivatives implicitly. إيجاد المشتقات للعلاقات الضمنية	Example 8.2	198
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		(13,14)	222

In exercises 9–14, find an equation of the tangent line.

13. $y - x^2y^2 = x - 1$ at $(1, 1)$

Find the slope to $y - x^2y^2 = x - 1$ at $(1, 1)$.

$$\frac{d}{dx}(y - x^2y^2) = \frac{d}{dx}(x - 1)$$

$$y' - 2xy^2 - x^2 2y \cdot y' = 1$$

$$y'(1 - x^2 2y) = 1 + 2xy^2$$

$$y' = \frac{1 + 2xy^2}{1 - 2x^2y}$$

At $(1, 1)$:

$$y' = \frac{1 + 2(1)(1)^2}{1 - 2(1)^2(1)} = \frac{3}{-1} = -3$$

The equation of the tangent line is

$y - 1 = -3(x - 1)$ or $y = -3x + 4$.

19	Find derivatives implicitly.	Example 8.2	198
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		(13,14)	222

In exercises 9–14, find an equation of the tangent line.

14. $y^2 + xe^y = 4 - x$ at $(2, 0)$

Implicitly differentiating:
$2yy' + e^y + xe^y y' = -1$, and
$$y' = \frac{-1 - e^y}{2y + xe^y}.$$
At $(2, 0)$ the slope is -1, and the equation of the tangent line is $y = -(x - 2)$.

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Question 20

Understand the Mean Value Theorem and use it in applications

Page 217, 220 & 223
Example 10.3
Exercise 43 – 46, 83 - 84

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Justin Dsouza
Teacher

20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
	التعرف على نظرية القيمة المتوسطة واستخدامها في التطبيقات	(43-46)	220
		(83,84)	223

Example 10.3 An Illustration of the Mean Value Theorem

Find a value of c satisfying the conclusion of the Mean Value Theorem for $f(x) = x^3 - x^2 - x + 1$ on the interval [0, 2].

Solution:

Notice that f is continuous on [0, 2] and differentiable on (0, 2). The Mean Value Theorem then says that there is a number c in (0, 2) for which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2 - 0} = 1.$$

To find this number c, we set

$$f'(c) = 3c^2 - 2c - 1 = 1$$

$$3c^2 - 2c - 2 = 0.$$

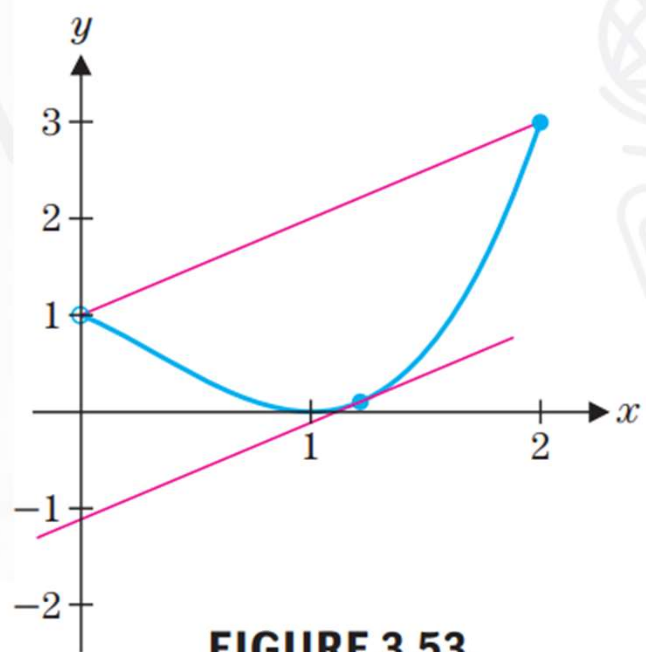

FIGURE 3.53
Mean Value Theorem

From the quadratic formula, we get $c = \frac{1 \pm \sqrt{7}}{3}$. In this case, only one of these, $c = \frac{1 + \sqrt{7}}{3}$, is in the interval (0, 2). In Figure 3.53, we show the graphs of $y = f(x)$, the secant line joining the endpoints of the portion of the curve on the interval [0, 2] and the tangent line at $x = \frac{1 + \sqrt{7}}{3}$.

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In exercises 43–46, explain why it is not valid to use the Mean Value Theorem. When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. In three of the four cases, show that there is no value of *c* that makes the conclusion of the theorem true. In the fourth case, find the value of *c*.

43. $f(x) = \dfrac{1}{x}, [-1, 1]$

$f(x) = 1/x$ on $[-1, 1]$. We easily see that $f(1) = 1$, $f(-1) = -1$, and $f'(x) = -1/x^2$. If we try to find the c in the interval $(-1, 1)$ for which

$$f'(c) = \frac{f(1) - f(1)}{1 - (-1)} = \frac{1 - (-1)}{1 - (-1)} = 1,$$

the equation would be $-1/c^2 = 1$ or $c^2 = -1$. There is of course no such c, and the explanation is that the function is not defined for $x = 0 \in (-1, 1)$ and so the function is not continuous.

The hypotheses for the Mean Value Theorem are not fulfilled.

20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
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In exercises 43–46, explain why it is not valid to use the Mean Value Theorem. When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. In three of the four cases, show that there is no value of c that makes the conclusion of the theorem true. In the fourth case, find the value of c.

44. $f(x) = \dfrac{1}{x^2}, [-1, 2]$

$f(x)$ is not continuous on $[-1, 2]$, and not differentiable on $(-1, 2)$. Can we find

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}\,?$$

$f'(x) = -\dfrac{2}{x^3} = -\dfrac{1}{4}$ when $x = 2$. This is not in $(-1, 2)$, so no c makes the conclusion of Mean Value Theorem true.

20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
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		(83,84)	223

In exercises 43–46, explain why it is not valid to use the Mean Value Theorem. When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. In three of the four cases, show that there is no value of *c* that makes the conclusion of the theorem true. In the fourth case, find the value of *c*.

45. $f(x) = \tan x, [0, \pi]$

$f(x) = \tan x$ on $[0, \pi]$, $f'(x) = \sec^2 x$. We know the tangent has a massive discontinuity at $x = \pi/2$, so as in #44, we should not be surprised if the Mean Value Theorem does not apply. As applied to the interval $[0, \pi]$ it would say

$$\sec^2 c = f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{\tan \pi - \tan 0}{\pi - 0} = 0.$$

But secant $= 1/\text{cosine}$ is never 0 in the interval $(-1, 1)$, so no such c exists.

20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
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In exercises 43–46, explain why it is not valid to use the Mean Value Theorem. When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. In three of the four cases, show that there is no value of *c* that makes the conclusion of the theorem true. In the fourth case, find the value of *c*.

46. $f(x) = x^{1/3}, [-1, 1]$

$f(x)$ is not differentiable on $(-1, 1)$. Can we find c with

$$f'(c) = \frac{f(1) - f(-1)}{1-(-1)} = \frac{1-(-1)}{2} = 1\,?$$

$f'(x) = \frac{1}{3}x^{-2/3} = 1$ when $x = \pm\left(\frac{1}{3}\right)^{3/2}$. These are both in $(-1, 1)$, so we can use either of these as c to make the conclusion of Mean Value Theorem true.

20	Understand the Mean Value Theorem and use it in applications.	Example 10.3	217
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In exercises 83 and 84, find a value of *c* as guaranteed by the Mean Value Theorem.

83. $f(x) = x^2 - 2x$ on the interval $[0, 2]$

$f(2) = 0 = f(0)$

If $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{0 - 0}{2} = 0$

then $2c - 2 = f'(c) = 0$ so $c = 1$.

84. $f(x) = x^3 - x$ on the interval $[0, 2]$

$f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the Mean Value Theorem applies. We need to find c so that

$f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{6 - 0}{2 - 0} = 3.$

$f'(x) = 3x^2 - 1 = 3$ when $x = \sqrt{4/3}$, so $c = 2\sqrt{3}/3$.

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