# Lesson 10-5 The Normal Distribution

التوزيع الطبيعي

Objectives:

**1** Find area under normal distribution curves.

**2** Find probabilities for normal distributions, and find data values given probabilities.

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## 1 The Normal Distribution

The probability distribution for a continuous variable is called a *continuous probability distribution*. The most widely used continuous probability distribution is called the **normal distribution**. The characteristics of the normal distribution are as follows.

### KeyConcept Characteristics of the Normal Distribution

ملامح التوزيع الطبيعي


المتوسط

- The graph of the curve is bell-shaped and symmetric with respect to the mean.

الوسيط المتوسط المنوال

- The mean, median, and mode are equal and located at the center.
- The curve is continuous.
- The curve approaches, but never touches, the $x$-axis.
- The total area under the curve is equal to 1 or 100%.

1 100%

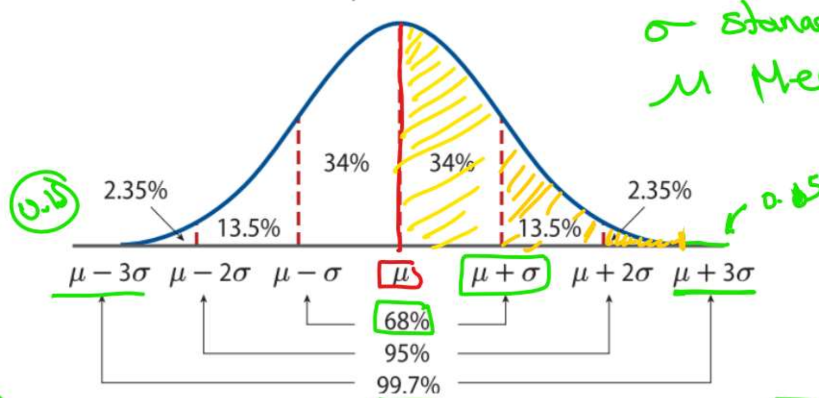
The area under the normal distribution curve between two data values represents the percent of data values that fall within that interval. The **empirical rule** can be used to describe areas under the normal curve over intervals that are one, two , or three standard deviations from the mean.

### KeyConcept The Empirical Rule

In a normal distribution with mean $\mu$ and standard deviation $\sigma$:

$\sigma$ Stanadrad deviation

$\mu$ Mean




$\mu = 150$
$\sigma = 5$

- approximately 68% of the data values fall between $\mu - \sigma$ and $\mu + \sigma$.
- approximately 95% of the data values fall between $\mu - 2\sigma$ and $\mu + 2\sigma$.
- approximately 99.7% of the data values fall between $\mu - 3\sigma$ and $\mu + 3\sigma$.

68% $\Rightarrow$ 150 − 5 and 150 + 5 $\Rightarrow$ 145 and 155

## Example 1 Use the Empirical Rule

Number of student = 880

**HEIGHT** The heights of the 880 students at Al-Sharq Secondary School are normally distributed with a mean of 168 cm and a standard deviation of 6 cm.

**a.** Approximately how many students are more than 180 cm tall?

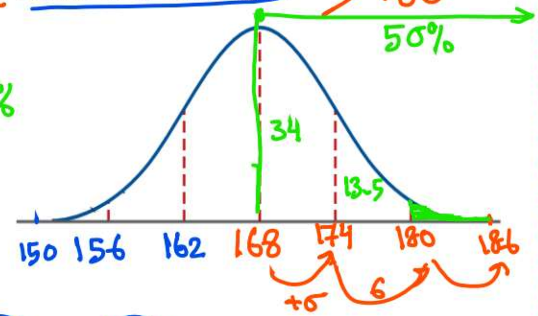
$* \ \mu = 168$ cm $* \ \sigma = 6$ cm

$P(X > 180) = 50\% - 34\% - 13.5\%$

$= 2.5\%$

Number of Student $= \frac{2.5}{100} \times 880$

$= 22$ Student



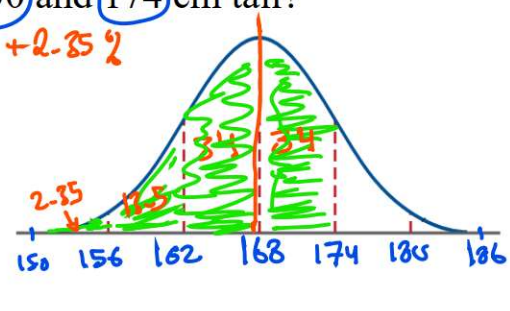

**b.** What percent of the students are between 150 and 174 cm tall?

$P(150 < X < 174) = 34\% + 34\% + 13.5\% + 2.35\%$

$= 83.85\%$

Number of Student $= \frac{83.85}{100} \times 880$

$= 737.88 = \boxed{738}$




## Guided Practice

**MANUFACTURING** A machine used to fill water bottles dispenses slightly different amounts into each bottle. Suppose the volume of water in 120 bottles is normally distributed with a mean of 1.1 liters and a standard deviation of 0.02 liter.

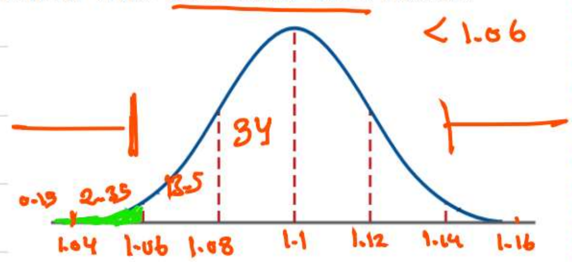
**A.** Approximately how many bottles of water are filled with less than 1.06 liters?

$\mu = 1.1 \qquad \sigma = 0.02$

$P(X < 1.06) = 2.5\%$

The number of bottles is

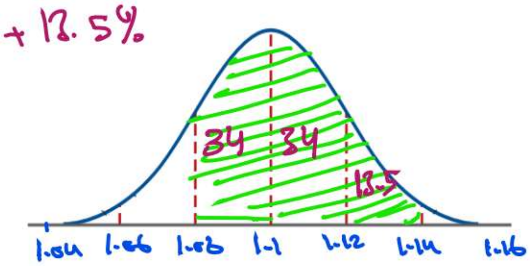
$\frac{2.5}{100} \times 120 = 3$ bottles




**B.** What percent of the bottles have between 1.08 and 1.14 liters?

$P(1.08 < X < 1.14) = 34\% + 34\% + 13.5\%$

$= 81.5\%$



## KeyConcept Formula for z-Values

The $z$-value for a data value in a set of data is given by $z = \frac{X - \mu}{\sigma}$, where $X$ is the data value, $\mu$ is the mean, and $\sigma$ is the standard deviation.

$\sigma z = X - \mu$

$\boxed{\sigma z + \mu = X}$

### Example 2 Find z-Values

$2.3(-1.73) + 48 = X$

**Find each of the following.**

| | |
|---|---|
| **a.** $z$ if $X = 24$, $\mu = 29$, and $\sigma = 4.2$<br>$z = \frac{X-\mu}{\sigma} = \frac{24-29}{4.2} = -1.19$ | ***b.*** $X$ if $z = -1.73$, $\mu = 48$, and $\sigma = 2.3$<br>$z = \frac{X-\mu}{\sigma}$<br>$-1.73 = \frac{X-48}{2.3}$<br>$-1.73(2.3) = X - 48$<br>$X = 44.021$ |

## GuidedPractice

$X = z\sigma + \mu$

**Find each of the following.**

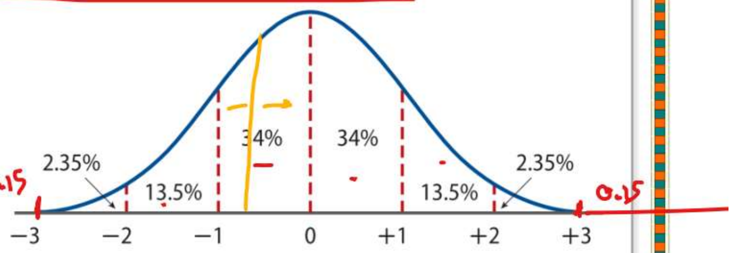
| | |
|---|---|
| **a.** $z$ if $X = 32$, $\mu = 28$, and $\sigma = 1.7$<br>$z = \frac{X-\mu}{\sigma} = \frac{32-28}{1.7} = 2.35$ | **b.** $X$ if $z = 2.15$, $\mu = 39$, and $\sigma = 0.4$<br>$z = \frac{X-\mu}{\sigma}$<br>$2.15 = \frac{X-39}{0.4}$<br>$\boxed{X = 39.86}$<br>$X = 2.15(0.4) + 39 =$ |
| **c.** $z$ if $X = 52$, $\mu = 43$, and $\sigma = 3.7$<br>$z = \frac{X-\mu}{\sigma} = \frac{52-43}{3.7}$<br>$z = 2.43$ | **d.** $X$ if $z = 1.7$, $\mu = 49$, and $\sigma = 4.1$<br>$z = \frac{X-\mu}{\sigma}$ or $X = z\sigma + \mu$<br>$1.7 = \frac{X-49}{4.1}$<br>$X = 55.97$<br>$= 1.7(4.1) + 49$<br>$= 55.97$ |
| **e.** $z$ if $X = 32$, $\mu = 38$, and $\sigma = 2.8$<br><br>Mr. Ali Abdalla | **f.** $X$ if $z = 2.5$, $\mu = 27$, and $\sigma = 0.4$ |

تدريب

The standard normal distribution is a normal distribution of $z$-values with a mean of 0 ( $\mu = 0$ ) and a standard deviation of 1 ( $\sigma = 1$ )

## KeyConcept Characteristics of the Standard Normal Distribution

- The total area under the curve is equal to 1 or 100%.
- Almost all of the area is between $z = -3$ and $z = 3$.
- The distribution is symmetric.
- The mean is 0, and the standard deviation is 1.
- The curve approaches, but never touches, the $x$-axis.




1.23

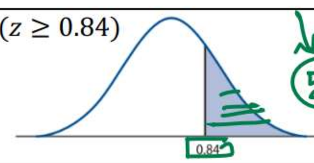
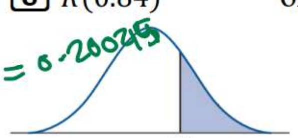
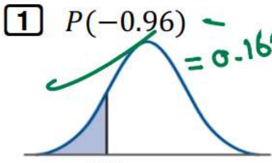
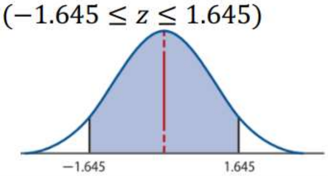
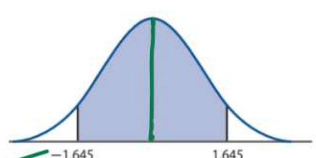
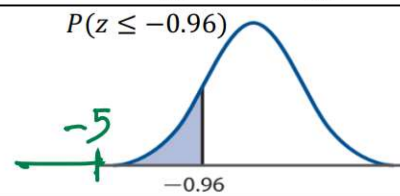
1.2 0.03

You can solve normal distribution problems by finding the $z$-value that corresponds to a given $X$-value, and then finding the approximate area under the standard normal curve. The corresponding area can be found by using a table of $z$-values that shows the area *to the left* of a given $z$-value. For example, the area under the curve to the left of a $z$-value of 1.12 is 0.8686, as shown.

**Standard Normal Distribution Table for Z = 0.00 to 3.59**
**AREA to the LEFT of the Z score**

| Z | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | 0.504 | 0.508 | 0.512 | 0.516 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.591 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.614 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.648 | 0.651 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.67 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.687 |
| 0.5 | 0.6915 | 0.695 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.719 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.754 |
| 0.7 | 0.758 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.785 |
| 0.8 | 0.7881 | 0.791 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.813 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.834 | 0.8365 | 0.838 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.862 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.877 | 0.879 | 0.881 | 0.883 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.898 | 0.8997 | 0.901 |

You can also find the area under the curve that corresponds to any z-value with a graphing calculator or ( CASIO Fx-991ES or Fx-991EX ). This method will be used for the remainder of this chapter.

| CASIO $fx$ -991ES | | CASIO $fx$ -991EX | |
|---|---|---|---|
| Mode | 1:COMP 2:CMPLX<br>3:STAT 4:BASE-N<br>5:EQN 6:MATRIX<br>7:TABLE 8:VECTOR<br>1 P left $\boxed{Z<}$ | MENU | 1:Calculate |
| 3 | 1:1-VAR 2:A+BX<br>3:_+CX² 4:ln X<br>5:e^X 6:A·B^X<br>7:A·X^B 8:1/X<br>3 R → Right $\boxed{Z>}$ | 7 | 1:Normal PD<br>2:Normal CD<br>3:Inverse Normal<br>4:Binomial PD |
| AC | STAT<br>0<br>2 Q $0<Z<\square$ | 2 | Normal CD<br>Lower:0<br>Upper:0<br>σ :1 |
| SHIFT 1<br>1:Type 2:Data<br>3:Sum 4:Var<br>5:Distr 6:MinMax | 5<br>1:P( 2:Q(<br>3:R( 4:▶t | Lower = 0.84<br>Upper = $\boxed{4}$<br>$P(z \geq 0.84)$<br><br>0.84<br>5 | |
| 3 $R(0.84)$<br>$= 0.20045$<br><br>0.84 | or 1 $P(-0.96)$<br>$= 0.16835$<br><br>−0.96 | Lower = −1.645<br>Upper = 1.645<br>$P(-1.645 \leq z \leq 1.645)$<br><br>−1.645 1.645 | |
| or 2<br>$Q(-1.645) + Q(1.645)$<br>0.90004 | <br>−1.645 1.645 | Lower = 4<br>Upper = −0.96<br>$P(z \leq -0.96)$<br>−5<br><br>−0.96 | |

### Using the Normal distribution table to determine the area corresponding to $z = 1.12$

- Locate the table of positive z-values.
- Locate in the first column the value 1.1 and in the first row the value 0.02.
- The area corresponding to the z-value of 1.12 is located at the intersection of the row and column, which is 0.8686.

## Example 3 Use the Standard Normal Distribution

n Number of days

**COMMUNICATION** The average number of phone calls received by a customer service representative each day during a 30-day month was 105 with a standard deviation of 12. Find the number of days with fewer than 110 phone calls. Assume that the number of calls is normally distributed.

$\boxed{\mu = 105}$

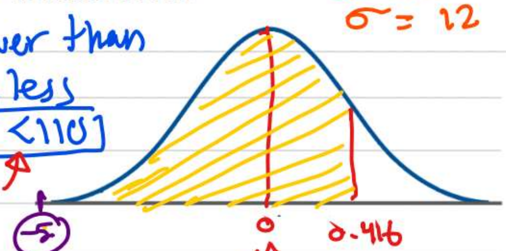
$\mu = 105$
$\sigma = 12$

$z = \frac{X - \mu}{\sigma} = \frac{110 - 105}{12} = 0.416$

fewer than = less $\boxed{X < 110}$

$P(X < 110) = P(Z < 0.416)$
$= 0.6613$

Number of day $= 0.6613 \times 30$
$= 19.8 \simeq 20$ days



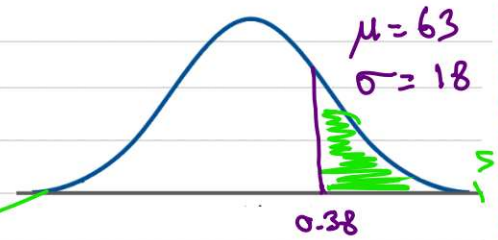

## Guided Practice

**BASKETBALL** The average number of points that a basketball team scored during a single season was 63 with a standard deviation of 18. If there were 15 games during the season, find the percentage of games in which the team scored more than 70 points. Assume that the number of points is normally distributed.

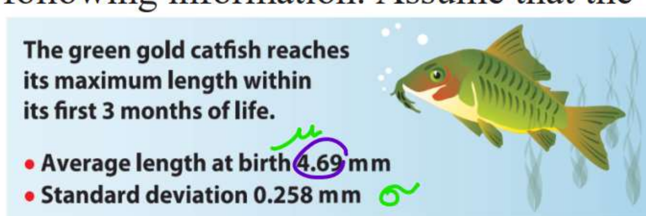
$\mu = 63$
$\sigma = 18$

$z = \frac{70 - 63}{18} = 0.38$

$P(X < 70) = 0.35197$
$= 35.197\%$




**ICHTHYOLOGY** As part of a science project, Mazen studied the growth rate of $\boxed{797}$ green gold catfish and found the following information. Assume that the data are normally distributed.




**a.** Determine the number of fish with a length less than 4.5 mm at birth. **184**

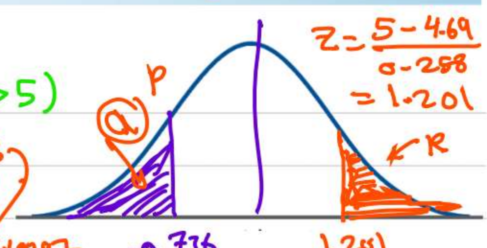
**b.** Determine the number of fish with a length greater than 5 mm at birth. **92**

a) $P(X < 4.5)$ $\quad z = \frac{4.5 - 4.69}{0.258} = -0.736$
$= 0.23087$

Number of fish $= 0.23087 \times 797$
$= 184$

b) $P(X > 5)$
$= 0.11488$
Number of fish $= \times 797$

$z = \frac{5 - 4.69}{0.258} = 1.201$



## Example 4 Find $z$-Values Corresponding to a Given Area

991 EX ARX

**Find the interval of $z$-values associated with each area.**

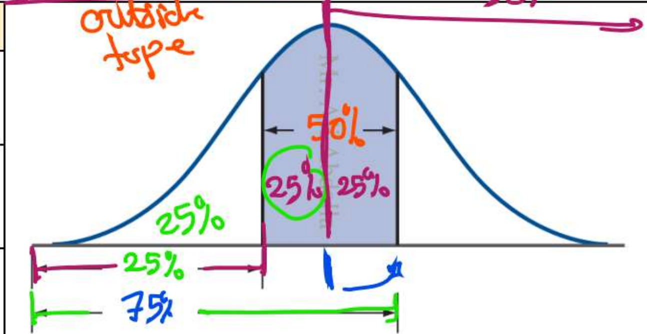
**a.** **Middle** 50% of the data.

Middle 50%

outside type

| CASIO $fx$-991 EX | |
|---|---|
| MENU | 1:Calculate |
| 7 | 1:Normal PD<br>2:Normal CD<br>3:Inverse Normal<br>4:Binomial PD |
| 3 | Inverse Normal<br>Area :0.25<br>$\sigma$ :1<br>$\mu$ :0 |
| Area = 0.25 $\Rightarrow$ $z = -0.674489579$<br>xInv=<br>-0.674489579 | Area = 0.75 $\Rightarrow$ $z = 0.674489579$<br>xInv=<br>0.674489579 |
| $-0.6745 < z < 0.6745$ | |

0.25
0.75

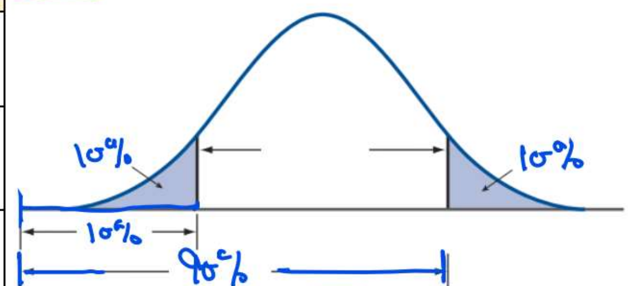



**b.** **The outside** 20% of the data

100%

| CASIO Fx-991EX | |
|---|---|
| MENU | 1:Calculate |
| 7 | 1:Normal PD<br>2:Normal CD<br>3:Inverse Normal<br>4:Binomial PD |
| 3 | Inverse Normal<br>Area :0.25<br>$\sigma$ :1<br>$\mu$ :0 |
| Area = 0.10 $\Rightarrow$ $z = -1.281551638$<br>xInv=<br>-1.281551638 | Area = 0.90 $\Rightarrow$ $z = 1.281551638$<br>xInv=<br>1.281551638 |
| $-1.2816 < z < 1.2816$ | |

0.10
0.90

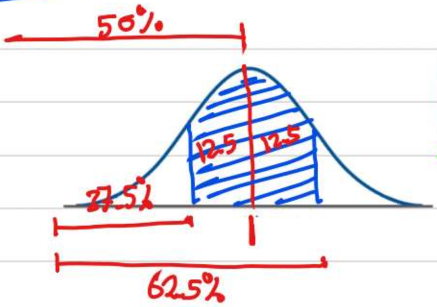



### GuidedPractice

**4A.** The middle 25% of the data

$A_1$ $\boxed{0.375}$ $\boxed{0.625}$

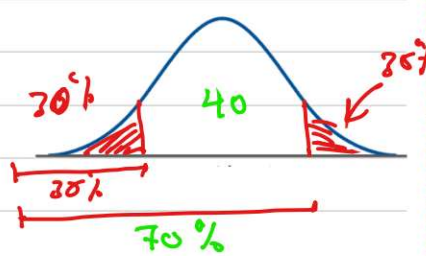
$-0.3186$
$0.3186$




$-0.3186 < Z < 0.3186$

**4B.** The outside 60% of the data

0.30
0.70
$-0.5244$
$0.5244$




$-0.5244 < Z < 0.5244$

## Example 5 Find Probabilities

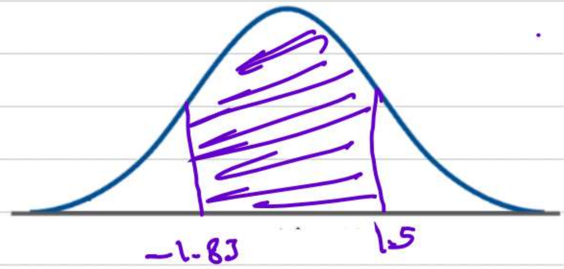
**METEOROLOGY** The temperatures for one month for a city in California are normally distributed with $\mu = 81°$ and $\sigma = 6°$. Find each probability and sketch the corresponding area under the curve.

$Z = \frac{X - \mu}{\sigma}$

**a.** $P(70° < \boxed{X} < 90°)$ $= 0.8996$

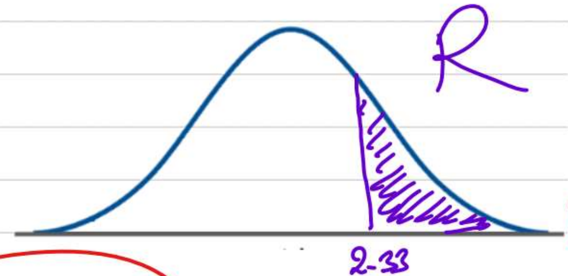
$z = \frac{70-81}{6} = -1.83$

$z = \frac{90-81}{6} = 1.5$




**a.** $P(X \geq 95°)$ $Z = \frac{95-81}{6} = 2.33$

$= 0.0098$

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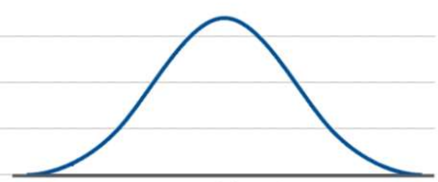
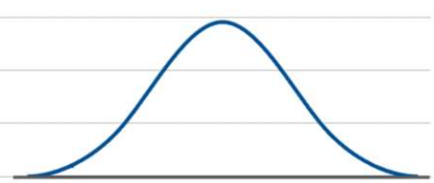
## GuidedPractice

**TESTING** The scores on a standardized test are normally distributed with μ = 72 and σ = 11. Find each probability and sketch the corresponding area under the curve.

**a.** $P(X < 89)$



**a.** $P(65 < X < 85)$



## Real-World Example 6 Find Intervals of Data

**COLLEGE** The scores for the entrance exam for a college's mathematics department is normally distributed with $\mu = 65$ and $\sigma = 8$.

invers Norm
991LGX

**a.** If Fatema wants to be in the top 20%, what score must she get?

first find z-vlaue

$z = 0.8416$

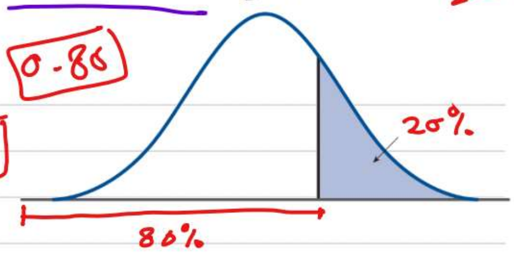
$z = \frac{x - \mu}{\sigma}$

$0.8416 = \frac{x - 65}{8}$

$\boxed{z\sigma + \mu = x}$

$0.8416(8) + 65 = x$

$x = 71.7 \simeq 72$




**b.** Fatema expects to earn a grade in the middle 90% of the distribution. What range of scores fall in this category?

$z = -1.64$, $z = 1.64$

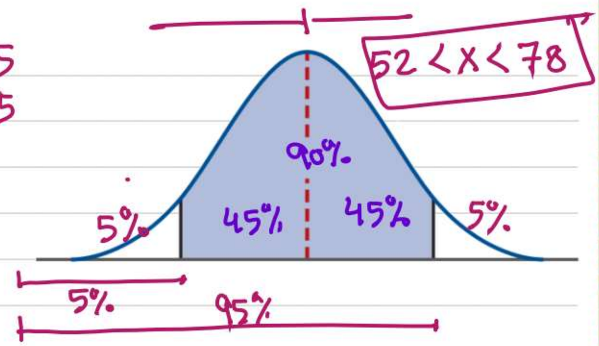
0.05
0.95

$-1.64 = \frac{x - 65}{8}$

$x = 51.88 \simeq 52$

$1.64 = \frac{x - 65}{8}$

$x = 78.12 \simeq 78$

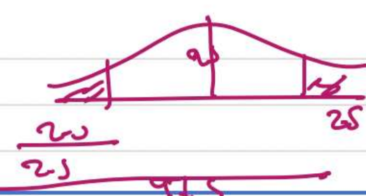



## GuidedPractice

**RESEARCH** As part of a medical study, a researcher selects a study group with a mean weight of 86 kg and a standard deviation of 5.5 kg. Assume that the weights are normally distributed.

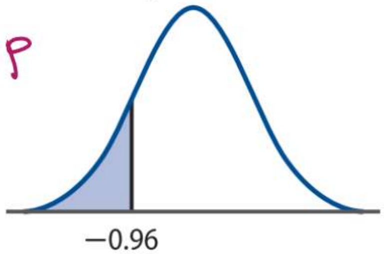
**A.** If the study will mainly focus on participants whose weights are in the middle 80% of the data set, what range of weights will this include?

**B.** If participants whose weights fall in the outside 5% of the distribution are contacted 2 weeks after the study, people in what weight range will be contacted?

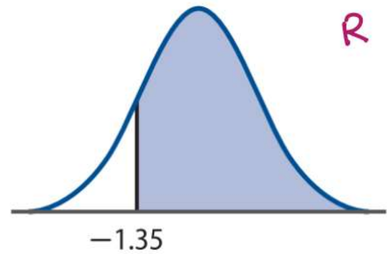


Find the area that corresponds to each shaded region.

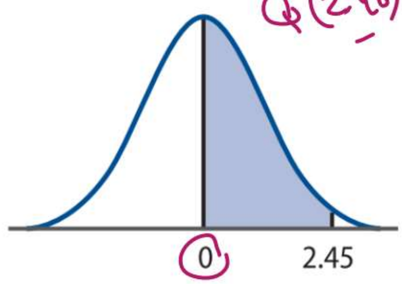
P



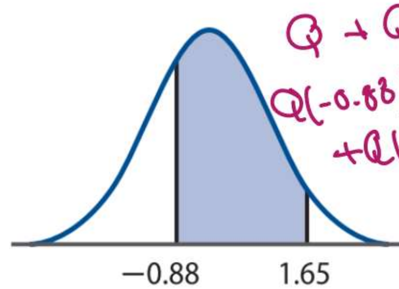

R




Q(2.45)




Q + Q
Q(-0.88)
+ Q(1.65)




The length of each song in a music collection is normally distributed with $\mu = 4.12$ minutes and $\sigma = 0.68$ minutes. Find the probability that a song selected from the collection at random is longer than 5 minutes.

**A** 10% **B** 19% **C** 39% **D** 89%

$P(X > 5) = P(Z > 1.29)$
$= 9.8\%$

1.29

$Z = \frac{X - \mu}{\sigma} = \frac{5 - 4.12}{0.68} = 1.29$

**PROJECTS** The scores on a science project for one class are normally distributed with $\mu = 78$ and $\sigma = 8$. Find each probability.

**a.** $P(X \geq 96)$ **b.** $P(60 < X < 85)$

$Z = \frac{96-78}{8}$
$= 2.25$

R

2.25

$P(Z \geq 2.25) = \ldots\ldots$

$Z = \frac{60 - 78}{8} = -2.25$

$Z = \frac{85 - 78}{8} = 0.875$ Q + Q

$\approx$ ---

-2.25 0.8

Suppose the test scores on a final exam are normally distributed with a mean of 74 and a standard deviation of 3. What is the probability that a randomly selected test has a score higher than 77? > 77

**A** 2.5% **B** 13.5% **C** 16% **D** 34%

$\mu = 74$ $\sigma = 3$

$Z = \frac{77 - 74}{3} = 1$

$P(Z=1) = 0.1586$
$= 15.86\%$

1

R