Unit 5

Applications of Differentiation

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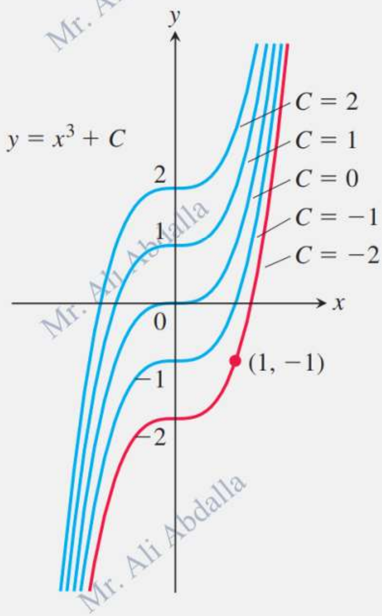
L 5.1 Antiderivative

1. Find the derivative of each function.

a) $F(x) = x^3 + 3$

b) $F(x) = x^3 + 7$

c) $F(x) = x^3 - 25$



Find the antiderivative of $f(x) = 3x^2$

Theorem 1.1

Suppose that F and G are both antiderivatives of f on an interval I. Then,

$$G(x) = F(x) + c,$$

for some constant c.

Definition 1.1

Let F be any antiderivative of f on an interval I. The indefinite integral of $f(x)$ (with respect to x) on I, is defined by

$$\int f(x)dx = F(x) + c$$

,where c is an arbitrary constant (the constant of integration).

The process of computing an integral is called **integration**. Here, $f(x)$ is called the **integrand** and the term dx identifies x as the **variable of integration**.

Evaluate:

2. $\int 3x^2\, dx$

3. $\int t^5\, dx$

Theorem 1.2 (Power Rule)

For any rational power $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Here, if $n < -1$, the interval I on which this is defined can be any interval that does not include $x = 0$.

By using Power rule Evaluate:

4. $\int x^9 \, dx$

5. $\int \frac{1}{x^5} \, dx$

6. $\int \sqrt{x} \, dx$

7. $\int \sqrt[3]{x^2} \, dx$

IMPORTANAT RULES

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad for \quad n \neq -1 \ (power\ rule)$$

$$\int \sin x \, dx = -\cos x + c \qquad \int \cos x \, dx = \sin x + c$$

$$\int \sec^2 x \, dx = \tan x + c \qquad \int \csc^2 x \, dx = -\cot x + c$$

$$\int \sec x \tan x \, dx = \sec x + c \qquad \int \csc x \cot x \, dx = -\csc x + c$$

$$\int e^x dx = e^x + c \qquad \int e^{-x} dx = -e^{-x} + c$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

$$\int \frac{1}{|x|\sqrt{1-x^2}} dx = \sec^{-1} x + c \qquad \int \frac{1}{x} dx = \ln|x| + c$$

تعميم لبعض قواعد التكامل يمكن استخدامها مباشرة:

A generalization of some rules of integration can be used directly

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$$

بالمثل مع باقي النسب المثلثية

$$\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int f'(x)\,a^{f(x)}\,dx = \frac{1}{\ln a}\,a^{f(x)} + c$$

Theorem 1.3

Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b,

$$\int [a\,f(x) + b\,g(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

8. $\int (3\cos x + 6x^5)\,dx$

9. $\int (\sin x + \sqrt{x})\,dx$

10. $\int \left(3e^x - \frac{2}{1+x^2}\right) dx$

11. $\int \left(\frac{x^{1/3} - 3}{x^{2/3}}\right) dx$

12. $\int \left(\dfrac{x + x^{3/4}}{x^{5/4}} \right) dx$

Theorem 1.4

For $x \neq 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

13. For any x for which $\tan x \neq 0$, Evaluate : $\frac{d}{dx}(\ln|\tan x|)$

Corollary 1.1	**Corollary 1.2**
In any interval not containing 0, $\int \frac{1}{x}\, dx = \ln\lvert x\rvert + c$	In any interval not containing 0, $\int \frac{f'(x)}{f(x)}\, dx = \ln\lvert f(x)\rvert + c$

Evaluate:

14. $\int \dfrac{4x}{x^2 + 4}\, dx$

15. $\int \dfrac{\sec^2 x}{\tan x}\, dx$

16. $\int \dfrac{\cos x}{\sin x}\, dx$

17. $\int \dfrac{x}{(x-1)(x+1)}\, dx$

18.	$\int (x-3)(x+4)\, dx$	**19.**	$\int \frac{x^3+1}{x}\, dx$
20.	$\int \frac{x^2+1}{\sqrt{x}}\, dx$		
21.	$\int \frac{4x+6}{x^2+3x}\, dx$		

Challenge

22. $\int \frac{x}{\sqrt{1-x^2}}\, dx$

23. $\int \frac{2x-1}{x^2+1}\, dx$

Finding the Position of a Falling Object Given Its Acceleration

24. If an object's downward acceleration is given by $y''(t) = -32ft/s^2$, find the position function $y(t)$. Assume that the initial velocity is $y'(0) = -100ft/s$ and the initial position is $y(0) = 100{,}000$ feet.

Find the derivative.

25. $\dfrac{d}{dx}(\ln|\sec x + \tan x|)$

26. $\dfrac{d}{dx}(\ln|\sin x - 2|)$

Find the general antiderivative.

25. $\int \frac{\cos x}{\sin x}\, dx$

26. $\int \left(2\cos x - \sqrt{e^{2x}}\right) dx$

27. $\int \frac{e^x}{e^x + 3}\, dx$

28. $\int \frac{e^x + 3}{e^x}\, dx$

29. $\int x^{1/4}\left(x^{5/4} - 4\right) dx$

30. $\int x^{2/3}\left(x^{-4/3} - 3\right) dx$

31. $\int \csc x\,(\sec x \tan x - \cot x)\, dx$

32. $\int \frac{\sin^2 x}{1 - \cos x} dx$

33. $\int \frac{1}{1 - \sin x} dx$

34. $\int \sin x \cos x \, dx$

35. $\int (\cos^2 x - \sin^2 x) \, dx$

36. $\int \sin^2 x \, dx$

37. $\displaystyle\int \frac{1-\sin 2x}{\sin x-\cos x}\,dx$

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38. $\displaystyle\int \frac{4}{1+\cos 2x}\,dx$

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39. $\displaystyle\int \frac{e^{\tan x}}{1-\sin^2 x}\,dx$

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40. $\displaystyle\int e^{x^2+\ln x}\,dx$

Challenge

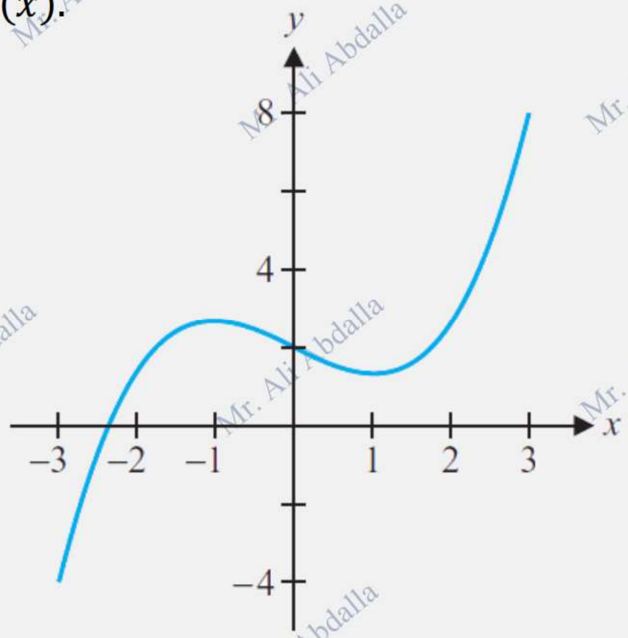
41. Evaluate :

$$\int \frac{x^3 + 4x}{x^4 + 1}\, dx$$

42. Find all functions satisfying the given conditions.

$$f'''(x) = \sin x - e^x$$

43. Sketch the graph of two functions $f(x)$ corresponding to the given graph of $y = f'(x)$.



44. Show that

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c \quad \text{and} \quad \int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1} x + c$$

Explain why this does not imply that $\cos^{-1} x = -\sin^{-1} x$. Find an equation relating $\cos^{-1} x$ and $\sin^{-1} x$

Challenge

45. **Evaluate :**

$$\int \sec x \ dx$$

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46. Determine the position function if the acceleration function is $a(t) = 3\sin t + 1$, the initial velocity is $v(0) = 0$ and the initial position is $s(0) = 4$.

47. Find the function $f(x)$ satisfying the given conditions.

a) $f'(x) = 4\cos x \quad , f(0) = 3$

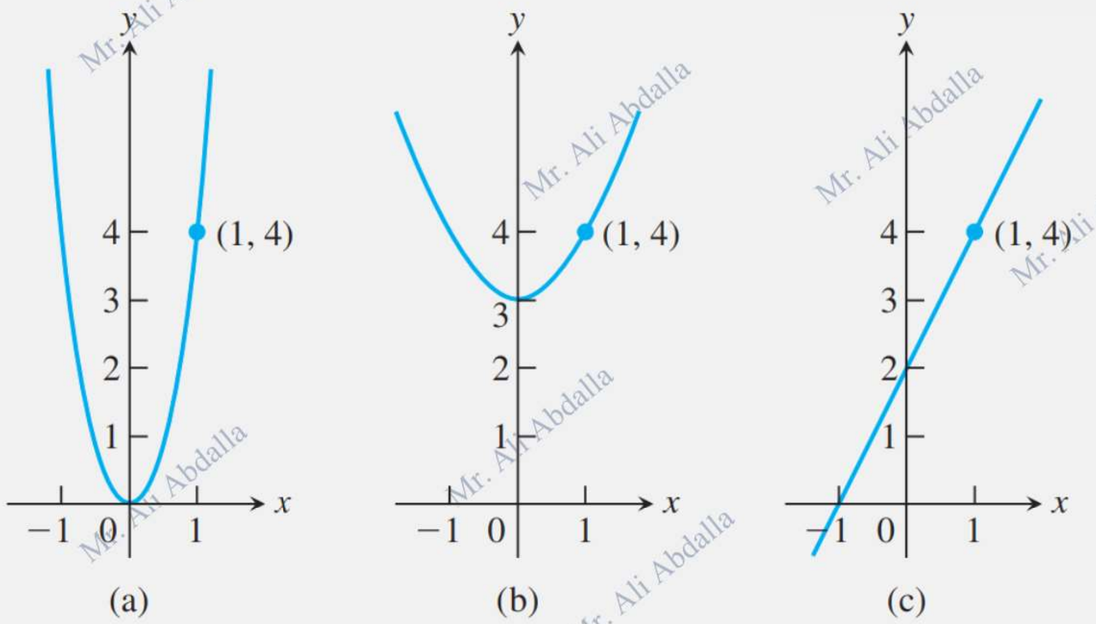
b) $f''(x) = 12x^2 + 2e^x \quad , f'(0) = 2 , \; f(0) = 3$

c) $f''(t) = 5 + 6t \quad , f(1) = 3 , \; f(-1) = -2$

48. Find a function $f(x)$ such that the point $(1,2)$ is on the graph of $y = f(x)$, the slope of the tangent line at $(1,2)$ is 3 and $f''(x) = x - 1$.

49. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$



50. Find the general antiderivative.

A) $\int 4\,\frac{\cos x}{\sin^2 x}\,dx$

B) $\int \frac{4}{\sqrt{1-x^2}}\,dx$

C) $\int (2x^{-1} + \sin x)\,dx$

D) $\int (3\cos x - \sin x)\,dx$

E) $\int 5\sec^2 x\,dx$

F) $\int \left(3\cos x - \frac{1}{x}\right)dx$

G) $\int 2\sec x \tan x\,dx$

I) $\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right)dx$

L 5.2 Sums and Sigma Notation

In general, for any real numbers $a_1, a_2, \ldots, a_n$, we have

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Write each of the following without sigma notation sign:

1) $\sum_{i=1}^{5} 3i - 1 =$

2) $\sum_{i=1}^{10} i^2 =$

3) $\sum_{i=1}^{5} i^2 - i =$

For questions (4-13): Write each of the following in summation notation

4) $3^3 + 4^3 + 5^3 + \cdots + 45^3 = \sum$

5) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{15} = \sum$

6) $\sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \cdots + \sqrt{50-1} = \sum$

7) $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 99 \times 100 = \sum$

8) $4 + 8 + 12 + \cdots + 400 = \sum$

9) $1 + 3 + 5 + \cdots + 101 = \sum$

10) $5 + 10 + 15 + \cdots + 205 = \sum$

11) $1.2 + 1.5 + 1.8 + \cdots + 31.2 = \sum$

12) $2 + 4 + 8 + 16 + \cdots + 4048 = \sum$

13) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2048} = \sum$

Sigma notation Rules and properties

for $n > 0$ if a, b and c are real numbers:

1) $\sum_{i=1}^{n} c = nc$ (Sum of constants)

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (Sum of the first n positive integers)

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (Sum of the squares of the first n positive integers).

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ (Sum of the cubes of the first n positive integers).

5) $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

6) $\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

7) $\sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r} \quad , \quad r \neq 1$ Geometric series

8) $\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r} \quad , \quad |r| < 1$ Geometric series

9) $\sum_{i=1}^{n} (ca_i \pm db_i) = c\sum_{i=1}^{n} a_i \pm d\sum_{i=1}^{n} b_i$ For any constants c and d

Use summation rules to compute the sum.

14) $\sum_{i=1}^{70} (3i-1) = \sum_{i=1}^{70} \qquad \sum_{i=1}^{70}$

15) $\sum_{i=1}^{40}(4-i^2) = \sum_{i=1}^{40} \qquad \sum_{i=1}^{40}$

16) $\sum_{n=1}^{100}(n^2-3n+2) = \sum_{n=1}^{100} \qquad \sum_{n=1}^{100} \qquad \sum_{n=1}^{100}$

17) $\sum_{k=0}^{20}(k^2+5) = \quad + \sum$

$= \quad + \sum \quad + \sum$

18) $\sum_{k=5}^{20}(k^2-3) = \sum \qquad - \sum$

Remember that: $\sum_{i=k}^{n} i$ The Number of terms $= n-k+1$

$\sum_{i=k}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{k-1} i$

$\sum_{i=0}^{n} f(i) = f(0) + \sum_{i=1}^{n} f(i)$

Computing a Sum of Function Values

19) Sum the values of $f(x) = x^2 + 3$
evaluated $at\ x = 0.1,\ x = 0.2, \ldots, x = 1.0$

20) Sum the values of $f(x) = 3x^2 - 4x + 2$
evaluated at $x = 1.05,\ x = 1.15,\ x = 1.25, \ldots,\ x = 2.95$

21) Compute sums of the form $\sum_{i=1}^{n} f(x_i)\,\Delta x$ for the given values of x_i .

$f(x) = x^2 + 4x$, $x = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$; $\Delta x = 0.2$; $n = 5$

22) Compute sums of the form $\sum_{i=1}^{n} f(x_i)\,\Delta x$ for the given values of x_i .

$f(x) = x^2 + 4x$, $x = 2,\ 4,\ 6, \ldots ,\ 100$

23) Compute sums of the form $\sum_{i=1}^{n} f(x_i)\,\Delta x$ for the given values of x_i .

$f(x) = x^3 + 4$; $x = 2.05,\ 2.15,\ 2.25,\ 2.35, \ldots ,\ 2.95$; $\Delta x = 0.1$; $n = 10$

24) Compute the sum and the limit of the sum as $n \to \infty$

A) $\displaystyle\sum_{i=1}^{n} \frac{3}{n}\left(\frac{i}{n}+2\right)$

B) $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2+2\left(\frac{i}{n}\right)\right]$

25) Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$, we have

as desired. So, the proposition is true for $n = 1$. Next, **assume** that

$$\sum_{i=1} i^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \text{Induction assumption}$$

for some integer $k \geq 1$.
In this case, we have by the induction assumption that for $n = k + 1$,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1} i^2 = \sum i^2 + \sum i^2$$

Translate each into summation notation and then compute the sum

26) The sum of the squares of the first 50 positive integers.

27) The square of the sum of the first 50 positive integers.

28) The sum of the square roots of the first 10 positive integers.

29) The square root of the sum of the first 10 positive integers

Use summation rules to compute the sum

30) $\sum_{i=3}^{30}[(i-3)^2+i-3]$

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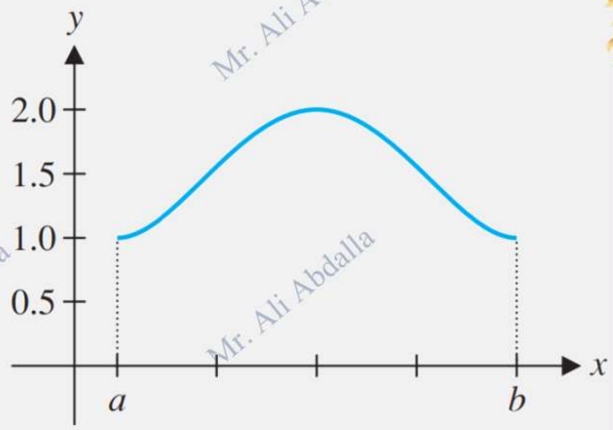
31) $\sum_{i=4}^{20}(i-3)(i+3)$

L 5.3 Area

Assume that $f(x) \geq 0$ and f is continuous on the interval $[a, b]$, as in Figure in the right.



- We start by dividing the interval $[a, b]$ into n equal pieces. This is called a regular partition of $[a, b]$.
- The width of each subinterval in the partition is then $\frac{b-a}{n}$, which we denote by Δx (meaning a small change in x).
- The points in the partition are denoted by
  $x_0 = a, \quad x_1 = x_0 + \Delta x, \quad x_2 = x_1 + \Delta x \quad , \dots \dots \dots \quad , x_n = b$
- In general, $\quad x_i = x_0 + i\Delta x, \quad$ for $\; i = 1, 2, \dots, n.$

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n} \quad , \quad x_0 = a$$

Then

$$\boxed{x_i = a + i\,\Delta x = a + \left(\frac{b-a}{n}\right) i}$$

Definition 3.2 (to find approximation Area)

Let $\{x_0, x_1, \dots, x_n\}$ be a regular partition of the interval $[a, b]$, with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for all i. Pick points $c_1, c_2, \dots, c_n$, where c_i is any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \dots, n$. (These are called evaluation points.) **The Riemann sum** for this partition and set of evaluation points is

$$\sum_{i=1}^{n} f(c_i)\,\Delta x \qquad \textbf{The Riemann sum}$$

Them the approximation area given by

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \Delta x \sum_{i=1}^{n} f(c_i) = \frac{b-a}{n} \sum_{i=1}^{n} f(c_i)$$

Definition 3.1 (to find exactly Area by limits)

For a function f defined on the interval $[a, b]$, if f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the area A under the curve $y = f(x)$ on $[a, b]$ is given by:

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Computing approximation Area

To approximation the area, we have three methods in this lesson, and we have other methods in lesson 5-7

To approximate the area under a curve by using n rectangles on the interval $[a, b]$ use the following summation.

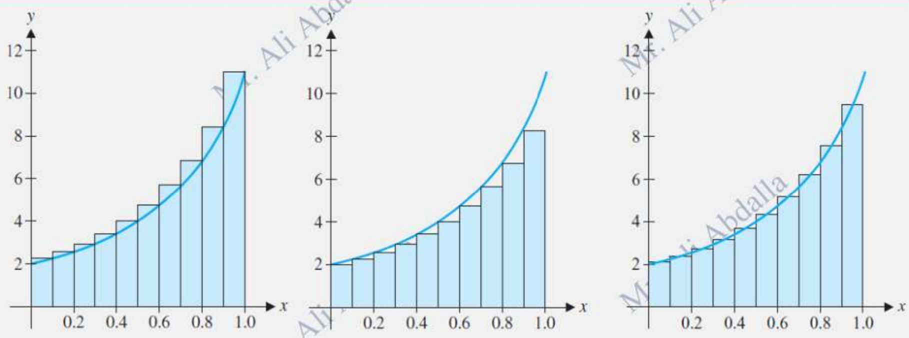
$$\sum_{i=1}^{n} f(c_i)\,\Delta x \qquad \text{و} \qquad \Delta x = x_i - x_{i-1} = \frac{b-a}{n}$$

Where:

1.	$c_i = x_i = a + \Delta x \cdot i$	The right-endpoint
2.	$c_i = x_{i-1} = a + \Delta x \cdot (i-1)$	The left-endpoint
3.	$c_i = \frac{1}{2}(x_{i-1} + x_i) = a + \Delta x \cdot \left(i - \frac{1}{2}\right)$	The midpoint

Exact area given by:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$



$c_i = x_i$ $\qquad$ $c_i = x_{i-1}$ $\qquad$ $c_i = \frac{1}{2}(x_{i-1} + x_i)$

===

We can find the area under a curve by using several methods, some method gives us approximate area and other give us exact area. We will summarize as following.

Area

- Approximate area
  - rectangles
    - The right-endpoint
    - The left-endpoint
    - The midpoint
  - Trapezoid
  - Simpson
- Exact arae
  - limit of The Riemann sum
  - integration

1) Approximate the area under the curve $y = f(x) = 2x - 2x^2$ on the interval [0, 1] by using 10 rectangles.

A) The right endpoint **B)** The left endpoint **C)** The midpoint

Then find the exact area.

Solution:

$\Delta x = \frac{1-0}{10} = 0.1$

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x_i)$											

The right-endpoint

The left-endpoint

The midpoint

c_i	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
$f(c_i)$										

Exact Area:

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\, \Delta x$$

2) Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.8	1.4	1.1	0.7	1.2	1.4	1.8	2.4	2.6

Left-endpoint

Right-endpoint

===

3) Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	2.0	2.2	1.6	1.4	1.6	2.0	2.2	2.4	2.0

Left-endpoint

Right-endpoint

4) Approximate the area under the curve of the function $f(x) = 2x + 3$ and x-axis on the interval $[1, 5]$ using n rectangles and the evaluation rules

(a) Left endpoint with $n = 20$ (b) Midpoint with $n = 10$

(c) Right endpoint with $n = 14$ (d) Exact Area

The Riemann sum: $\sum_{i=1}^{n} f(c_i)\,\Delta x$ و $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$

(a) left endpoint with $n = 20$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b) midpoint with $n = 10$

(c) right endpoint with $n = 14$

(d) Exact Area

5) Find exact area under the curve of the function $f(x) = 4x - x^2$ and x-axis on the interval $[0, 4]$ by using limits of The Riemann sum.

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \qquad , \quad c_i = x_i = a + \frac{b-a}{n}\, i$$

6) approximate the area under the curve on the given interval using n rectangles and the evaluation rules

(a) left endpoint (b) midpoint (c) right endpoint.

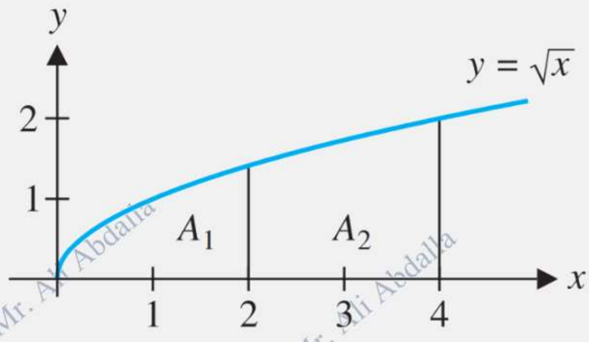
A) $y = \sqrt{x+2}$ on $[1,4]$; $n = 16$

B) $y = e^{-2x}$ on $[-1,1]$; $n = 16$

C) $y = \cos x$ on $\left[0, \frac{\pi}{2}\right]$; $n = 50$

7) In the figure, which area equals $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{2}\left(\sqrt{1+\frac{i}{n}}\right)\left(\frac{2}{n}\right)$

Ans. A_2



Use right endpoint

Consider interval $[2, 4]$, then $\Delta x = \frac{2}{n}$.

Use right endpoints as evaluation points, $x_i = \left(2+\frac{2i}{n}\right)$.

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\sqrt{2+\frac{2i}{n}}\right)\frac{2}{n}\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{2}\left(\sqrt{1+\frac{i}{n}}\right)\frac{2}{n}\right]$$

Hence,

$$A_2 = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{2}\left(\sqrt{1+\frac{i}{n}}\right)\frac{2}{n}\right].$$

8) In the figure, which area equals $\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n}}\left(\sqrt{1+2i}\right)\left(\frac{2}{n}\right)$

Ans. A_1

Use midpoint

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$c_i = a + \left(i-\frac{1}{2}\right)\Delta x$$

$$c_i = 0 + \frac{2}{n}\left(i-\frac{1}{2}\right) = \frac{1}{n}(2i-1)$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{1}{n}(2i-1)}\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{1}{n}}\sqrt{2i-1}\left(\frac{2}{n}\right)$$

Let $i = k+1$ when $i = 1 \Rightarrow k = 0$

when $i = n \Rightarrow k = n-1$

$$A = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n}}\sqrt{2(k+1)-1}\left(\frac{2}{n}\right)$$

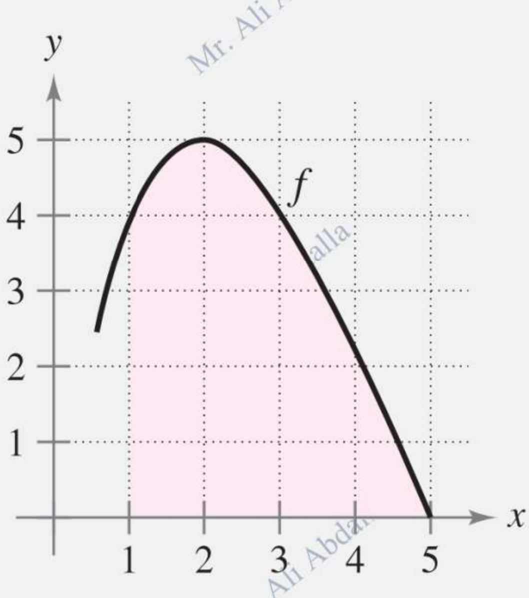
$$A = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n}}\sqrt{2k+1}\left(\frac{2}{n}\right)$$

$$\Rightarrow A_1 = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n}}\sqrt{2i+1}\left(\frac{2}{n}\right)$$

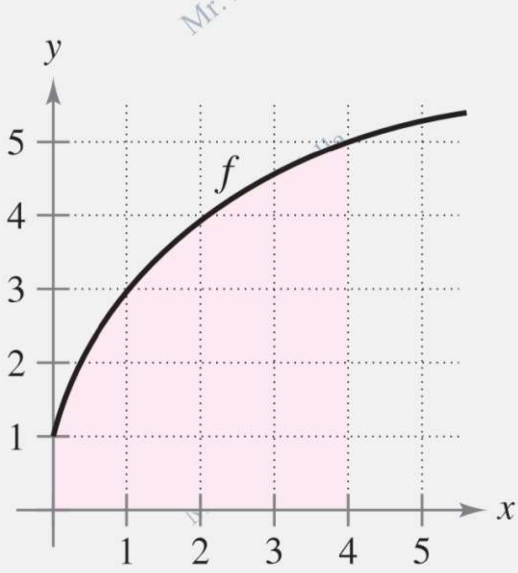
On $[0, 2]$

In Exercises 9 and 10, use the given graph to estimate the left Riemann sum for the given interval with the stated number of subdivisions.

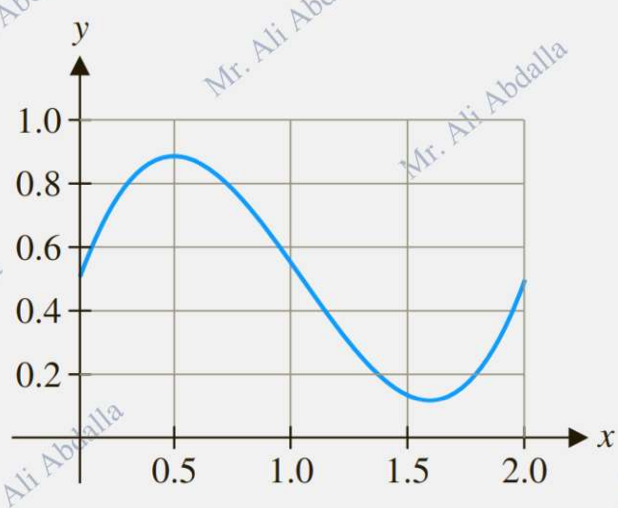
9) $[1,5], \quad n = 4$



10) $[0,4], \quad n = 4$



11) $[0,2], \quad n = 4$



L 5.4 The Definite Integral

Definition 4.1

For any function f defined on $[a, b]$, the **definite integral** of f from a to b is:

Upper limit

Lower limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x$$

whenever the limit exists and is the same for every choice of evaluation point, $c_1, c_2, \ldots, c_n$. When the limit exists, we say that f is **integrable** on $[a, b]$.

Properties of Definite Integral

Definition

$\int_a^a f(x)\,dx = 0$ $\qquad$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Constant Multiple

$\int_a^b c\,dx = c(b-a)$ $\qquad$ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

Sum and Difference

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Integrals of Symmetric Functions

If f is even $f(-x) = f(x)$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

If f is odd $f(-x) = -f(x)$, then $\int_{-a}^a f(x)\,dx = 0$

Comparison Property

If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge 0$

If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Rewrite each limit as definite integral form:

1) $$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x \quad , \quad [1,4]$$

2) $$\lim_{n\to\infty}\sum_{i=1}^{n}(2c_i^2-3c_i-2)\,\Delta x \quad , \quad [0,10]$$

3) $$\lim_{n\to\infty}\sum_{i=1}^{n}(2\cos c_i-\sin c_i)\,\Delta x \quad , \quad [0,\pi]$$

Rewrite each definite integral as limit form:

4) $$\int_0^1 (x^2-3x+2)dx$$

5) $$\int_0^3 \left(\frac{1}{1+x^2}\right)dx$$

Evaluate the integral by computing the limit of Riemann sums:

6) $\int_0^1 2x \; dx$

7) $\int_1^2 2x \; dx$

$$= \int \qquad dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

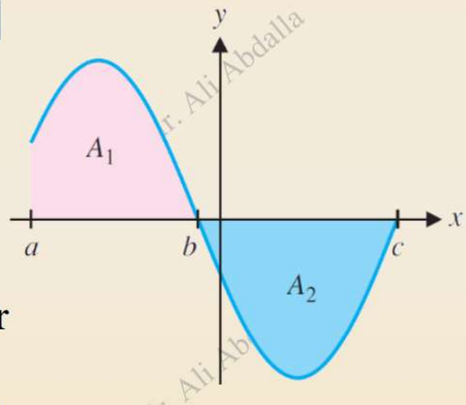
8) $\int_0^3 (x^2 - 3x) \; dx$

9) $\int_1^3 (x^2 - 3x) \; dx$

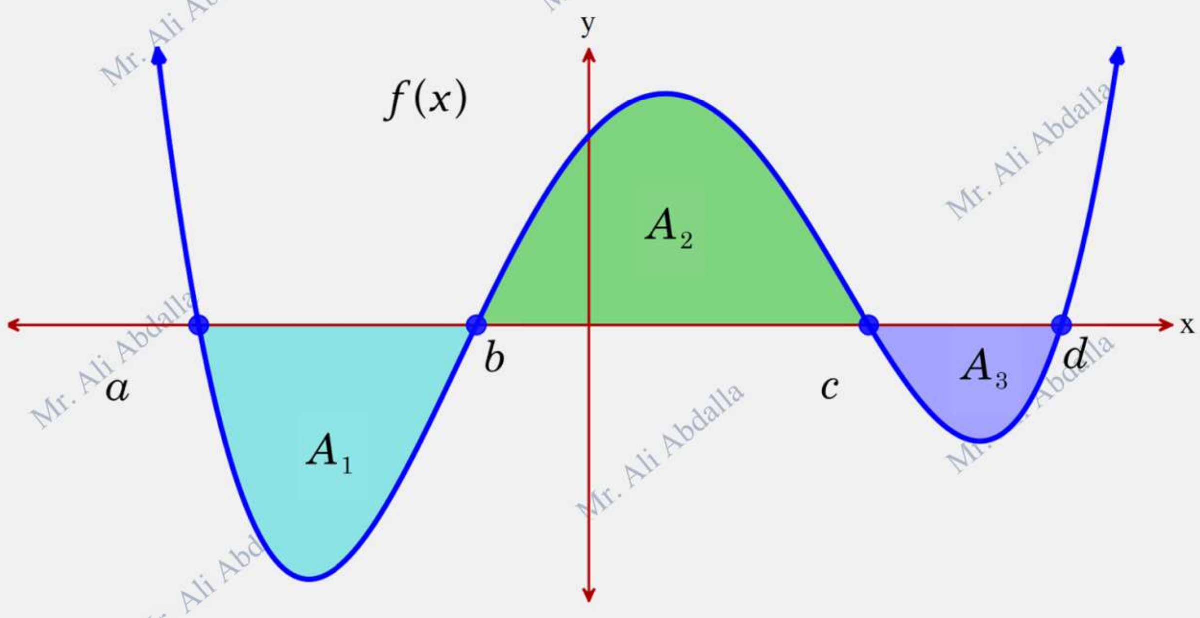
$$= \int \qquad dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

Definition 4.2

Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and A_1 is the area bounded between the curve $y = f(x)$ and the x-axis for $a \leq x \leq b$. Further, suppose that $f(x) \leq 0$ on the interval $[b, c]$ and A_2 is the area bounded between the curve $y = f(x)$ and the x-axis for $b \leq x \leq c$. The **signed area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 - A_2$, and the **total area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 + A_2$.

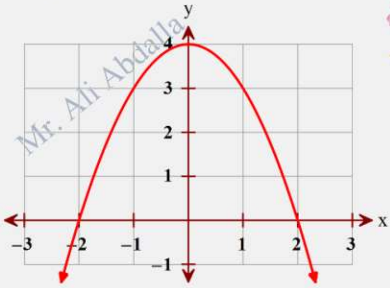


This means **signed area** is the difference between any areas lying above the x-axis and any areas lying below the x-axis, while the **total area** is the sum of the area bounded between the curve $y = f(x)$ and the x-axis.

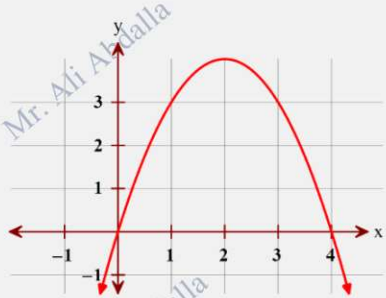


Write the given (**total**) area as an integral or sum of integrals.

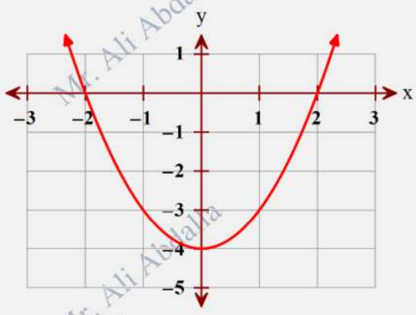
10) The area above the x-axis and below $y = 4 - x^2$



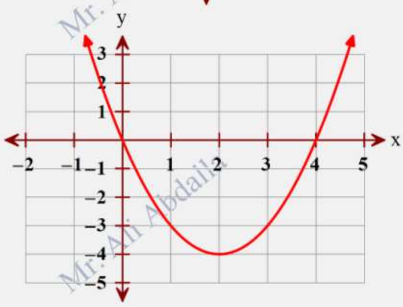
11) The area above the x-axis and below $y = 4x - x^2$



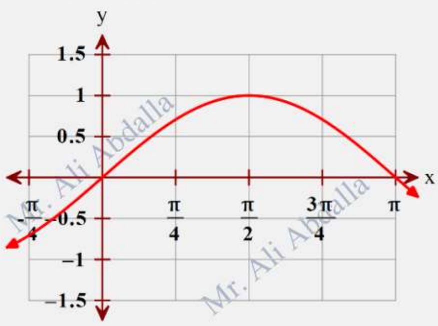
12) The area below the x-axis and above $y = x^2 - 4$



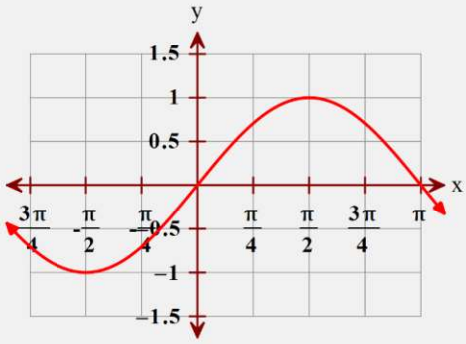
13) The area below the x-axis and above $y = x^2 - 4x$



14) The area between $y = \sin x$ and the x-axis for $0 \le x \le \pi$



15) The area between $y = \sin x$ and the x-axis for $-\frac{\pi}{2} \le x \le \frac{\pi}{4}$.



Theorem 4.2

If f and g are integrable on $[a, b]$, then the following are true.

(i) For any constant c and d ,

$$\int_a^b [c\, f(x) + d\, g(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$

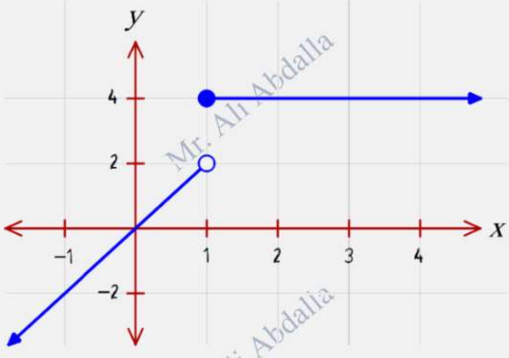
(ii) For any c in $[a, b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

(iii) $\int_b^a f(x)\,dx = -\int_a^b f(x)dx$ (iv) $\int_a^a f(x)\,dx = 0$

Integration at point

16) Evaluate $\int_0^4 f(x)\,dx$, where $f(x)$ is defined by $f(x) = \begin{cases} 2x & , x < 1 \\ 4 & , x \geq 1 \end{cases}$



17) Write the expression as a single integral.

A) $\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$

B) $\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$

C) $\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx$

D) $\int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx$

18) Assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ find

A) $\displaystyle\int_1^3 [f(x) + g(x)]\,dx$

B) $\displaystyle\int_1^3 [2f(x) - g(x)]\,dx$

C) $\displaystyle\int_1^3 [f(x) - g(x)]\,dx$

D) $\displaystyle\int_1^3 [4\,g(x) - 3f(x)]\,dx$

Theorem 4.3

Suppose that $g(x) \le f(x)$ for all $x \in [a, b]$ and that f and g are integrable on $[a, b]$. Then,

$$\int_a^b g(x)\,dx \le \int_a^b f(x)dx$$

Average Value of a Function

Average Value of a Function

$$f_{ave} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)\,dx$$

19) Compute the average value of $f(x) = x^2 + 2x$ on the interval $[0, 1]$.

20) Compute the average value of $f(x) = \sin x$ on the interval $[0, \pi]$

Squeeze property

Let f any continuous function defined on $[a, b]$ and it has a minimum m, and a maximum M, on $[a, b]$, so that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

This inequality used to approximate the value of the integration $\int_a^b f(x)\,dx$

Use the Integral Mean Value Theorem to estimate the value of the integral:

21) $\displaystyle\int_{\pi/3}^{\pi/2} 3\cos x^2\,dx$

22) $\displaystyle\int_0^{1/2} e^{-x^2}\,dx$

Theorem 4.4 (Integral Mean Value Theorem)

If f is continuous on $[a, b]$, then there is a number $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Find a value of c that satisfies the conclusion of the Integral Mean Value Theorem

23) $\int_0^2 3x^2\,dx \quad (= 8)$

24) $\int_{-1}^1 (x^2 - 2x)\,dx \quad \left(= \frac{2}{3}\right)$

25) Find bounds for $\int_0^2 x^2 e^{-\sqrt{x}}\,dx$

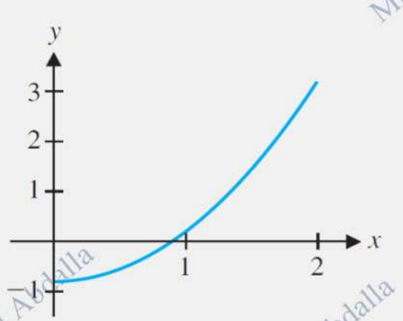
26) Express this limit as an integral:

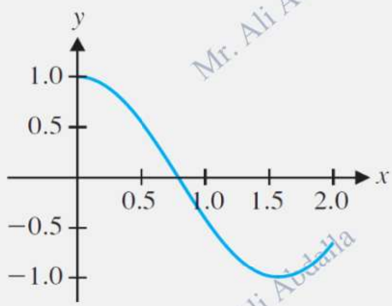
$$\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right]$$

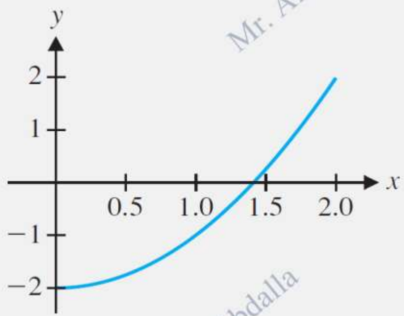
27) Show that the value of $\int_0^{\pi}\sqrt{1+\sin x}\,dx$ is between π and $\sqrt{2}\,\pi$

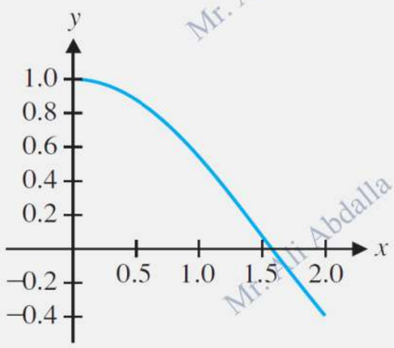
28) Show that the value of $\int_0^{1}3x^2\sqrt{1+x^2}\,dx$ is between 1 and $\sqrt{2}$

29) Use the graph to determine whether $\int_0^2 f(x)\,dx$ is positive or negative.



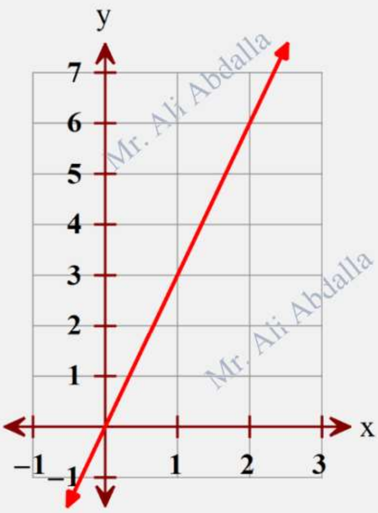


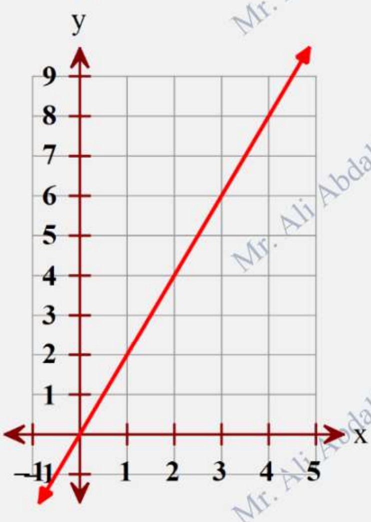


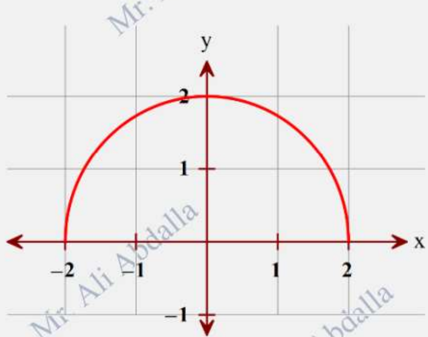


30) Use a geometric formula to compute the integral:

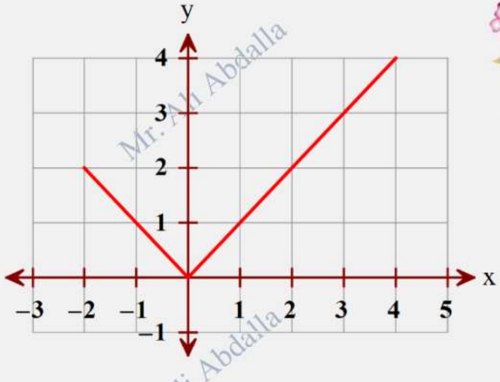
A $\int_0^2 3x\,dx$ B $\int_1^4 2x\,dx$ C $\int_0^2 \sqrt{4-x^2}\,dx$



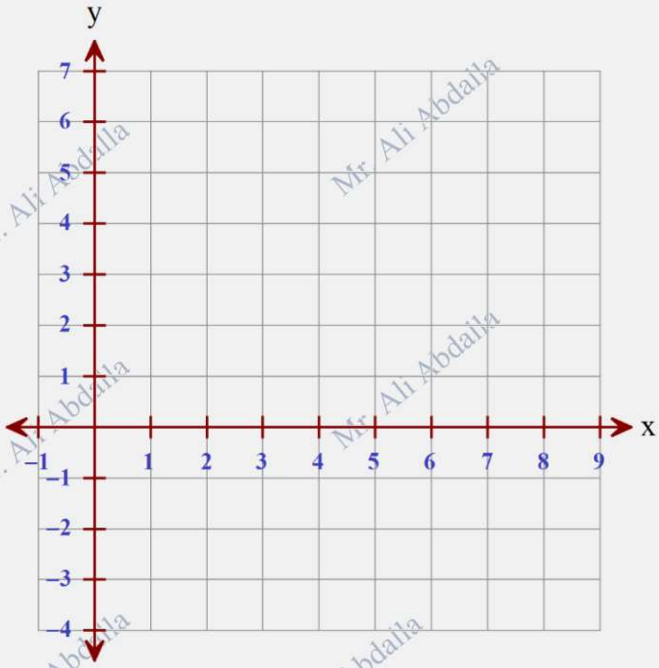




D $\int_{-1}^{2} |x|\, dx$



31) Graph the function $f(x) = \begin{cases} 3x-3, & 0 \le x < 3 \\ 6 & , 3 \le x < 5 \\ 21-3x, & 5 \le x \le 8 \end{cases}$ then find $\int_0^8 f(x)\, dx$



32) If $\int_3^7 f(x)\, dx = 5$ and $\int_3^7 g(x)\, dx = 3$, then all of the following must be true except:

(A) $\int_3^7 f(x)g(x)dx = 15$ **(B)** $\int_3^7 [f(x) + g(x)]dx = 8$ **(C)** $\int_3^7 2f(x)\, dx = 10$

(D) $\int_3^7 [f(x) - g(x)]dx = 2$ **(E)** $\int_7^3 [g(x) - f(x)]dx = 2$

33) The expression $\frac{1}{20}\left(\sqrt{\frac{1}{20}}+\sqrt{\frac{2}{20}}+\sqrt{\frac{3}{20}}+\cdots+\sqrt{\frac{20}{20}}\right)$

is a Riemann sum approximation for:

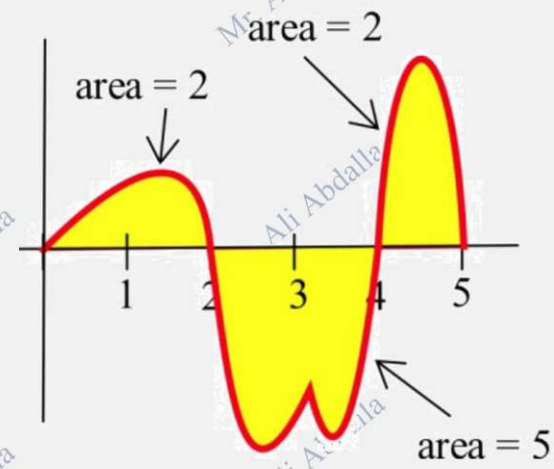
A) $\int_0^1 \sqrt{\frac{x}{20}}\,dx$ C) $\frac{1}{20}\int_0^1 \sqrt{\frac{x}{20}}\,dx$

B) $\int_0^1 \sqrt{x}\,dx$ D) $\frac{1}{20}\int_0^1 \sqrt{x}\,dx$

---

34) Use the graph to calculate: $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$ and $\int_0^5 f(x)\,dx$

$\int_a^b f(x)dx$ = area above − area below



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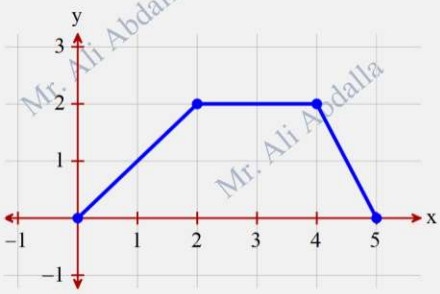
35) Let $A(x)$ represent the area bounded by the graph and the horizontal axis and vertical lines at $t = 0$ and $t = x$ for the graph shown.

Evaluate $A(x)$ for $x = 1, 2, 3, 4,$ and 5.



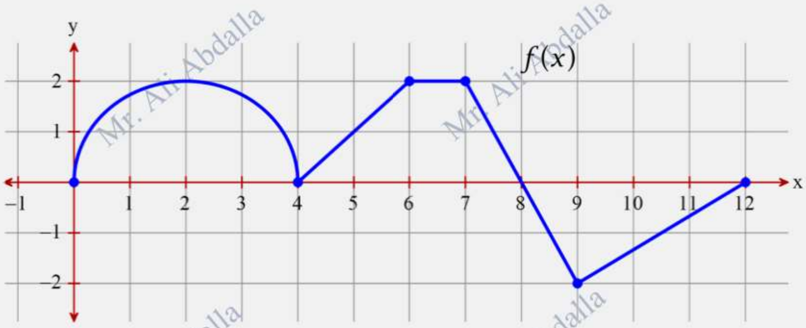
36) Use the graph to find

A) $\int_0^{12} f(x)\,dx$

B) $\int_0^{4} f(x)\,dx =$

C) $\int_{12}^{8} 3\,f(x)\,dx =$

D) $\int_0^{12} |f(x)|\,dx =$



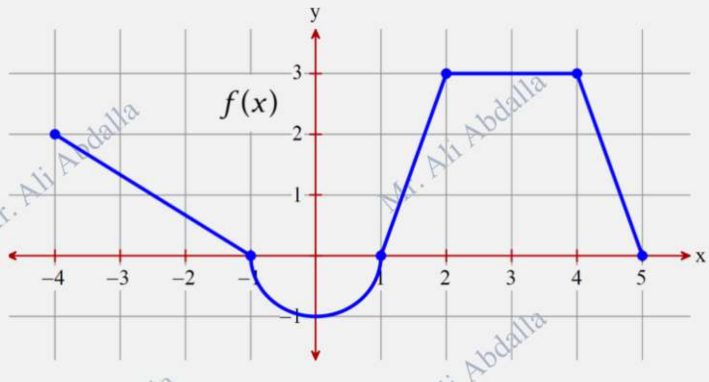
37) Use the graph to find

A) $\int_{-4}^{-1} f(x)\,dx$

B) $\int_{2}^{1} f(x)\,dx =$

C) $\int_{-4}^{1} |f(x)|\,dx =$

D) $\int_{-4}^{5} f(x)\,dx =$



38) Let f and g be continuous functions that produce the following definite integral values. $\int_1^2 f(x)\,dx = -2$, $\int_1^6 f(x)\,dx = 4$, $\int_1^6 g(x)\,dx = 8$

Find the following:

A) $\int_2^2 g(x)\,dx$	B) $\int_6^1 g(x)\,dx$	C) $3\int_1^2 f(x)\,dx$
D) $\int_2^6 f(x)\,dx$	E) $\int_1^6 [f(x) - g(x)]\,dx$	F) $\int_1^6 [3f(x) - g(x)]\,dx$
G) $\int_1^6 \lvert f(x) - g(x)\rvert\,dx$ Cannot be determined	H) $\left\lvert \int_1^6 [f(x) - g(x)]\,dx \right\rvert$	

Theorem 5.1

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Evaluate

1) $\int_1^3 (3x^2 + 2x)\,dx$

5) $\int_0^2 \frac{e^{2x} - 2e^{3x}}{e^{3x}}\,dx$

2) $\int_0^{\frac{\pi}{2}} \sin x\,dx$

6) $\int_0^1 (\sin^2 x + \cos^2 x)\,dx$

3) $\int_1^x (2t + 3)\,dt$

7) $\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$

4) $\int_{-1}^1 e^{-2x}\,dx$

8) $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

9) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

10) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos x\, dx$

11) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 2x\, dx$

12) $\displaystyle\int_0^{1} \frac{2x+1}{1+x^2}\,dx$

The Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$ and $F(x) = \int_0^x f(t)\,dt$,
then $F'(x) = f(x)$, on $[a,b]$.

In general: if $g(x) = \int_0^{u(x)} f(t)\,dt$, then $g'(x) = f\big(u(x)\big)\ u'(x)$ **or**

$$g'(x) = \frac{d}{dx}\int_0^{u(x)} f(t)\,dt = f\big(u(x)\big)\ u'(x) \quad \text{on } [a,b]$$

In general: if $g(x) = \int_{v(x)}^{u(x)} f(t)\,dt$, on $[a,b]$ then

$$g'(x) = \frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f\big(u(x)\big)\ u'(x) - f\big(v(x)\big)\, v'(x)$$

Examples:

13) For $f(x) = \int_{1}^{x} (t^2 + t - 1)\, dt$ compute $f'(x)$

14) For $f(x) = \int_{3}^{x^2} \sin t\, dt$ compute $f'(x)$

15) For $f(x) = \int_{3x}^{x^2} \sqrt{t^2 + 2t - 4}\, dt$ compute $f'(x)$

16) For $f(x) = \int_{3x}^{\sin x} \sqrt{t^2 + 2t - 4}\, dt$ compute $f'(x)$

17) For the function $F(x) = \int_{4}^{x^2} \ln(t^3 + 4)\, dt$ find an equation of the tangent line at $x = 2$.

18) Identify all local extrema of $f(x) = \int_0^x (t^2 - 3t + 2)\ dt$

19) If $f(x) = \int_1^x \sqrt{5t^2 - 1}\ dt$ Find the value of $f'(1)$ and $f(1)$

20) If $\int_0^x f(t)\, dt = x(\ln x - 1)$ Find the value of $f(e^2)$.

21) For $f(x) = \int_{\sqrt{2x}}^1 e^{t^2}\, dt$ compute $f'(x)$

22) For $f(x) = \int_{x}^{\sin^{-1} x} \sin t \, dt$ where $x \in (-1,1)$. Compute $f'(x)$

23) For $f(x) = \int_{\cos x}^{\sin x} \sqrt{1 - t^2} \, dt$ Where $x \in \left[0, \frac{\pi}{2}\right]$. Show that $f'(x) = 1$

24) For $f(x) = x + \int_{0}^{\tan x} \frac{1}{1 + t^2} dt$ Show that $f'(x) = 2$

25) $\int_0^8 \left(\frac{1}{\sqrt{1+x}}\right) dx$

A. 1 **B.** 2 **C.** 4 **D.** $\frac{3}{2}$ **E.** 6

---

26) $\int_0^1 \left(\frac{x^2}{1+x^2}\right) dx$

A. $\frac{4-\pi}{4}$ **B.** $\ln 2$ **C.** 0 **D.** $\frac{1}{2}\ln 2$ **E.** $\frac{4+\pi}{4}$

---

27) If $F(x) = \int_0^x e^{-t^2}\, dt$ Then $F'(x)$

A. $2xe^{-x^2}$ **B.** $-2xe^{-x^2}$ **C.** e^{-x^2}

D. $e^{-x^2} - 1$ **E.** $\frac{e^{-x^2+1}}{-x^2+1} - e$

---

28) If $f(x) = \int_0^x \frac{1}{\sqrt{t^3+1}} dt.$ Which of the following is **FALSE**?

A. $f(0) = 0$ **B.** $f(1) > 0$ **C.** $f(-1) > 0$

D. $f'(1) = \frac{1}{\sqrt{2}}$ **E.** f is continuous at for all $x \geq 0$

29) If F and f are continuous functions such that $F'(x) = f(x)$ for all x, then $\int_a^b f(x)\,dx$ is

A. $F'(a) - F'(b)$ **B.** $F'(b) - F'(a)$ **C.** $F(a) - F(b)$

D. $F(b) - F(a)$ **E.** none of the above

---

30) $\int_0^1 (x+1)e^{x^2+2x}\,dx$

A. $\frac{e^3}{2}$ **B.** $\frac{e^3-1}{2}$ **C.** $\frac{e^3-e}{2}$ **D.** $e^3 - 1$ **E.** $e^3 - e$

---

31) Given $f(x) = \begin{cases} x+1, & x < 0 \\ \cos \pi x, & x \geq 0 \end{cases}$ Then $\int_{-1}^1 f(x)\,dx$

A. $\frac{1}{2} + \frac{1}{\pi}$ **B.** $-\frac{1}{2}$ **C.** $\frac{1}{2} - \frac{1}{\pi}$ **D.** $\frac{1}{2}$ **E.** $-\frac{1}{2} + \pi$

---

32) $\int_{-1}^2 \frac{|x|}{x}\,dx$

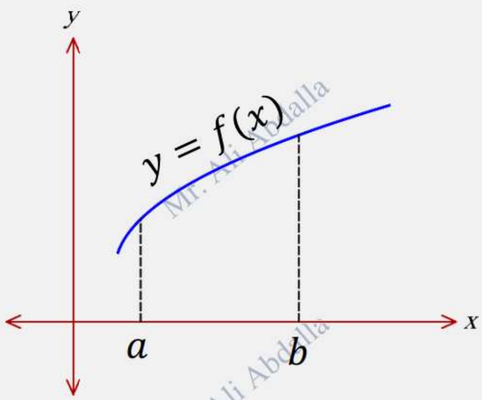
A. -3 **B.** 1 **C.** 2 **D.** 3 **E.** non-existent

33) If n is a known positive integer, for what value of k is: $\int_1^k x^{n-1}\,dx = \frac{1}{n}$

A. 0 **B.** $\left(\frac{2}{n}\right)^{\frac{1}{n}}$ **C.** $\left(\frac{2n-1}{n}\right)^{\frac{1}{n}}$ **D.** $2^{\frac{1}{n}}$ **E.** 2^n

---

34) If f is the continuous, strictly increasing function on the interval $a \le x \le b$ as shown on the right, which of the following must be true?



I. $\int_a^b f(x)\,dx < f(b)(b-a)$

II. $\int_a^b f(x)\,dx > f(a)(b-a)$

III. $\int_a^b f(x)\,dx = f(c)(b-a)$ For some numbers c such that $a < c < b$

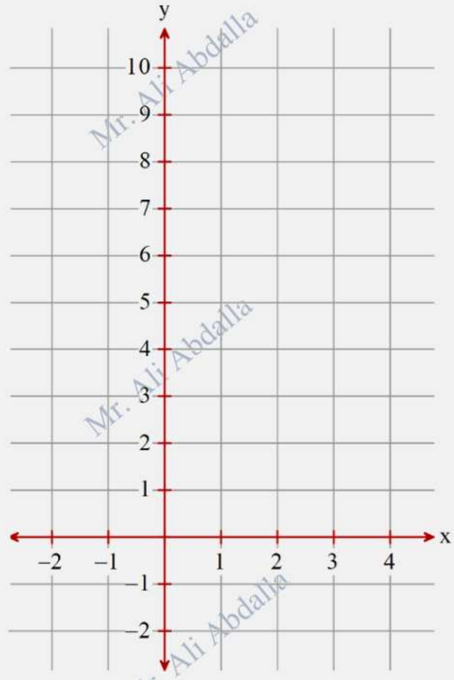
A. I only **B.** II only **C.** III only
D. I and II only **E.** I, II and III

---

35) If $f(x) = 2x|x+1|$ find $\int_{-2}^{3} f(x)\,dx$

36) If $f(x) = 2[\![x+3]\!]$, Where $[\![\ ]\!]$ the greatest integer function.

Find $\int_{-1}^{3} f(x)\ dx$



---

37) By using the table in the right

Find $h'(2)$ if

$$h(x) = \int_{2}^{g(x)} f(t)\, dt$$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	10	−2
1	−5	−8	5	1
2	15	−1	1	3

Let $p(t)$ represent the function of population

Let $b(t)$ represent the birth rate and $a(t)$ represent the death rate

Rate of change in population is $p'(t)$

Then $p'(t) = b(t) - a(t)$

The rate of change over one year (12 months) is

$$\int_{0}^{12} p'(t)\, dt$$

38) Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 410 - 0.3t$ organisms per month and the death rate is given by $a(t) = 390 + 0.2t$ organisms per month.

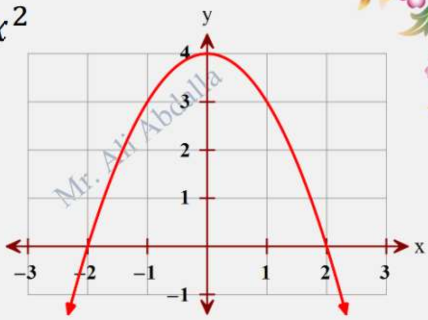
A) Explain why $\int_0^{12} [b(t) - a(t)]\, dt$ represents the net change in population in the first 12 months.

B) Determine for which values of t it is true that $b(t) > a(t)$.

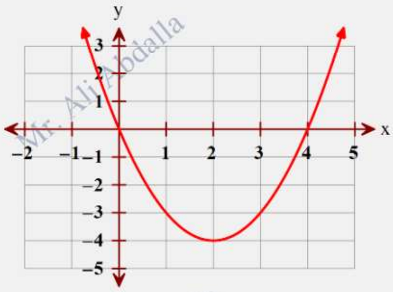
C) At which times is the population increasing? Decreasing?

D) Determine the time at which the population reaches a maximum.

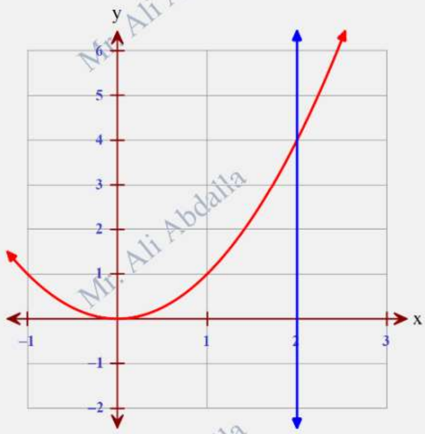
39) The area above the x-axis and below $y = 4 - x^2$



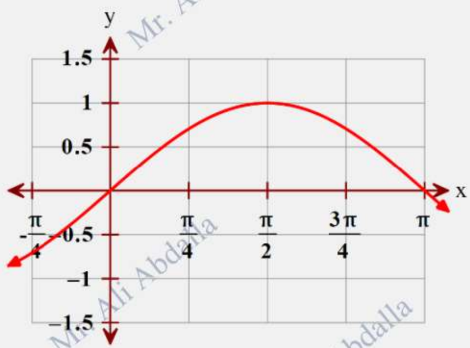
40) The area below the x-axis and above $y = x^2 - 4x$



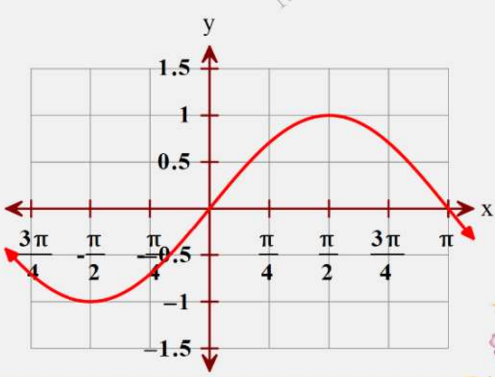
41) The area of the region bounded by $y = x^2$, $x = 2$ and the x-axis.



42) The area between $y = \sin x$ and the x-axis for $0 \leq x \leq \pi$



43) The area between $y = \sin x$ and the x-axis for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{4}$

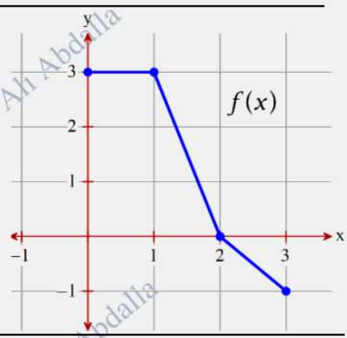


44) If $\quad g(x) = \int_{\pi}^{\pi x} \cos t^2 \, dt \quad$ then $\; g'(x) =$

A) $\sin(\pi^2 x^2)$ B) $\pi x \sin(\pi^2 x^2)$ C) $\pi \cos(\pi^2 x^2)$
D) $\pi x \cos(\pi^2 x^2)$ E) $\cos(\pi^2 x^2)$

---

45) The graph of f is given, and g is an antiderivative of f.
If $g(3) = 6$, find $g(0)$



(A) 1
(B) 2
(C) 4
(D) 5

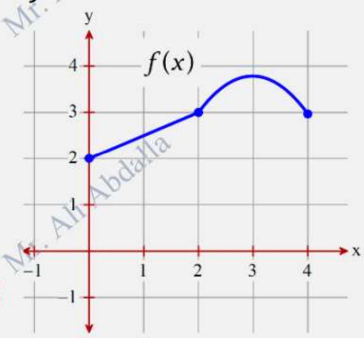
$$\int_0^3 f(x)\,dx = g(x)|_0^3 = g(3) - g(0)$$

$$\tfrac{1}{2}(1+2)(3) - \tfrac{1}{2}(1)(1) = 6 - g(0)$$

$$g(0) = 2$$

---

46) The graph of f is given, and $F(x)$ is an antiderivative of f.
If $\int_2^4 f(x)\,dx = 7.5$, find $F(4) - F(0)$.



A) 1.5
B) 7.5
C) 12.5
D) 18.5

$$\int_0^4 f(x)\,dx = F(4) - F(0)$$

$$\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = F(4) - F(0)$$

$$\tfrac{1}{2}(2+3)(2) + 7.5 = F(4) - F(0) \Rightarrow F(4) - F(0) = 12.5$$

---

47) Find $\int_{-2}^{2} f(x)\,dx$ if $\; f(x) = \begin{cases} 2x^2, & -2 \le x \le 0 \\ \sin 2x, & 0 < x \le 2 \end{cases}$

A) 4.507 **B)** 5.403 **C)** 6.161 **D)** 10.667

---

48) If $h(x) = \int_0^{2x} (e^{\cos t} - 1)\,dt$ on $(3,6)$. On which interval(s) is h decreasing?

A) (3.927, 5.498) **B)** (5.498, 6) **C)** (3, 4.712)
D) Always decreasing on (3, 6) **E)** Never decreasing on (3, 6)

49) $\lim\limits_{h\to 0} \dfrac{\int_1^{1+h} \sqrt{x^5+8}\,dx}{h}$

(A) 0 (B) 1 (C) 3 (D) $2\sqrt{2}$ (E) does not exist

---

50) $\lim\limits_{x\to 1} \dfrac{\int_1^{x} e^{t^2}\,dx}{x^2-1}$

(A) 0 (B) 1 (C) $\frac{e}{2}$ (D) e (E) does not exist

---

51) The graph of g′ , the first derivative of the function g, consists of a semicircle of radius 2, and two line segments, as shown in the figure below. If $g(0) = 1$, what is $g(3)$?

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

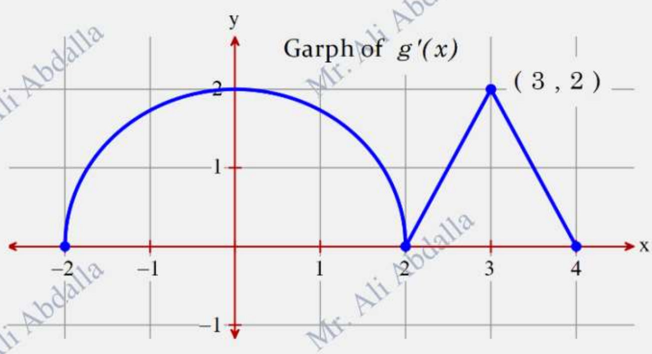
Fundamental theorem of calculus Part I

$$\int_0^3 g'(x)\,dx = g(3) - g(0)$$

$$\underbrace{\int_0^2 g'(x)\,dx}_{\text{Area of a quarter circle}} + \underbrace{\int_2^3 g'(x)\,dx}_{\text{Area of a triangle}} = g(3) - g(0)$$

$$\frac{\pi(2)^2}{4} + \frac{1}{2}(1)(2) = g(3) - 1$$

$$g(3) = \pi + 2$$

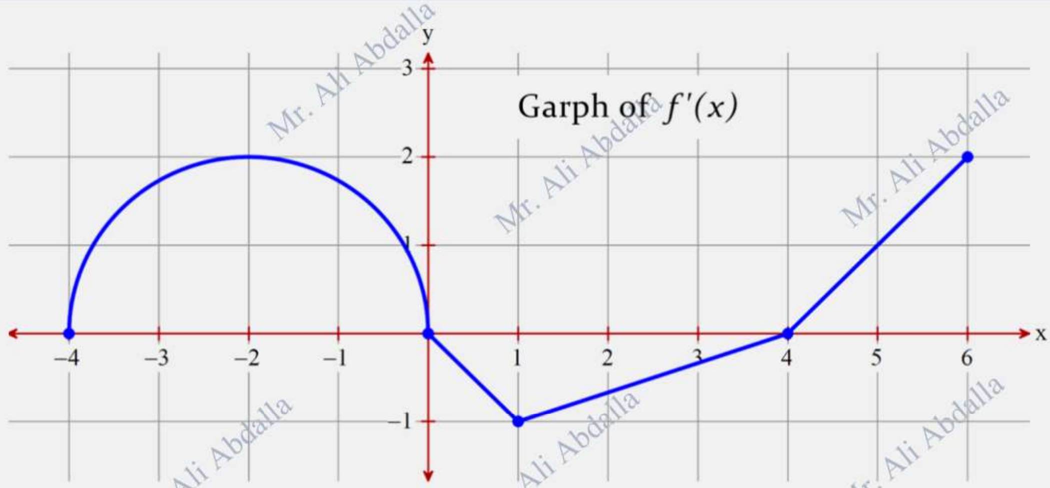


(A) $\pi + 1$ (B) $\pi + 2$
(C) $2\pi + 1$ (D) $2\pi + 2$

---

52) Let f be the function given by $f(x) = \int_1^x (3t - 6t^2)\,dt$.
What is the x-coordinate of the point of inflection of the graph of f?

A) $-\frac{1}{4}$ B) $\frac{1}{4}$ C) 0 D) $\frac{1}{2}$



53) The figure above represents the function f' a continuous function, the derivative of f over the interval $[-4, 6]$ and satisfies $f(0) = 4$.
The graph of f' consists of three line segments and a semi-circle.

A) Find the value of $f(-4)$.

$$\int_{-4}^{0} f'(x)\,dx = f(0) - f(-4) \Rightarrow f(-4) = \underbrace{f(0)}_{f(0)=4 \text{ given}} - \underbrace{\int_{-4}^{0} f'(x)\,dx}_{\text{Area of semi-circle}}$$

$$f(-4) = 4 - \frac{\pi(2)^2}{2} = 4 - 2\pi$$

B) On what interval(s) is f decreasing and concave up? Justify your answer.

$f(x)$ is decreasing and concave up on the interval $(1,4)$ because $f'(x)$ is negative and increasing.

C) State all x-values where $f(x)$ has a horizontal tangent on the open interval $(-4, 6)$. Explain whether f has a relative minimum, relative maximum, or neither at each of those x-values.

At $x = 0$ and $x = 4$, $f(x)$ has a horizontal tangent.
At $x = 0$, $f(x)$ has a relative maximum because f' changes from positive to negative.
At $x = 4$, $f(x)$ has a relative minimum because f' changes from negative to positive.

D) Evaluate $\int_2^3 f''(2x)\,dx$

$$\int_2^3 f''(2x)\,dx = \frac{1}{2} f'(2x)\Big|_2^3 = \frac{1}{2}\big(f'(6) - f'(4)\big) = \frac{1}{2}(2 - 0) = 1$$

E) Critical numbers of the function $f(x)$

Critical numbers is the x-coordinates when the graph of $f'(x)$ intersect with x-axis and the end points if included in its domain.
then the critical numbers are: $x = -4$, $x = 0$, $x = 4, x = 6$

x	0	3	6	9
$f(x)$	10	8	5	2

54) Let $g(x)$ be a twice-differentiable function defined by a differentiable function f, such that $g(x) = 2x + \int_1^{x^2} f(t)\,dt$. Selected values of $f(x)$ are given in the table above.

A) Use a Left Riemann sum using the subintervals indicated by the table to approximate $g(3)$.

$$g(3) = 2(3) + \int_1^9 f(x)\,dx = 6 + \Delta x[f(x_0) + f(x_1) + f(x_2)]$$

$$g(3) \approx 6 + 3[\ 10 + 8 + 5\ ] = 75$$

B) Find $g'(3)$.

$$g(x) = 2x + \int_1^{x^2} f(t)\,dt \Rightarrow g'(x) = 2 + 2x\,f(x^2)$$

$$\Rightarrow g'(3) = 2 + 2(3)f(9) = 2 + 6(2)$$

$$\Rightarrow g'(3) = 14$$

C) Using the data in the table, estimate $f'(4)$.

$$f'(4) \approx \frac{f(6) - f(3)}{6 - 3} = \frac{5 - 8}{3} = -1$$

D) Explain why there must be a value of c, on $1 < x < 9$ such that $f(c) = 4$

Since $g(x)$ is twice differentiable, $g'(x)$ is continuous therefore IVT applies. There must be a value of c, on $1 < x < 9$, such that $g'(c) = f(c) = 4$ because $f(6) > 4 > f(9)$.

---

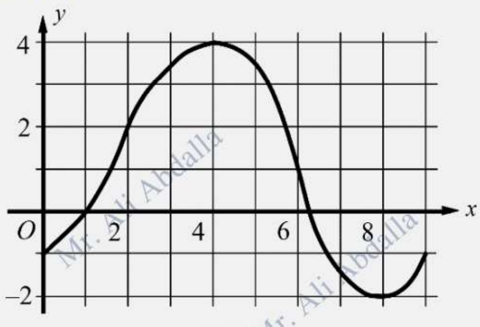
55) The graph of the function g , shown in the right figure, has horizontal tangents at $x = 4$ and $x = 8$. If $f(x) = \int_0^{\sqrt{x}} g(t)\,dt$, what is the value of $f'(4)$?

A) 0

B) $\frac{1}{2}$

C) $\frac{3}{4}$

D) $\frac{3}{2}$



graph of g

L 5.6 Integration by Substitution

In this section, we significantly expand our ability to compute antiderivatives by developing a useful technique called **integration by substitution.**

Integration by substitution consists of the following general steps:

- ☑ Choose a new variable u: a common choice is the innermost expression or "inside" term of a composition of functions.
- ☑ Compute $du = \frac{du}{dx}dx$.
- ☑ Replace all terms in the original integrand with expressions involving u and du.
- ☑ Evaluate the resulting (u) integral. If you still cannot evaluate the integral, you may need to try a different choice of u.
- ☑ Replace each occurrence of u in the antiderivative with the corresponding expression in x.

How to choose the correct substitution to be u:

A) Inside brackets like $(x^2+1)^5$ use $u = x^2+1$

B) Inside roots like $\sqrt{3x^2+2}$ use $u = 3x^2+2$

C) The exponents like: e^{x^2+2} use $u = x^2+2$

D) The angles like $\sin(3x^4)$ use $u = 3x^4$

And maybe other substitution

Evaluate

1) $\int x^2(x^3+2)^{100}\,dx$

2) $\int (3x+4)^7\,dx$

3) $\int x \sin x^2 \, dx$

4) $\int (3\tan x + 4)^5 \sec^2 x \, dx$

5) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

6) $\int \frac{x^5}{1+x^6} \, dx$

7) $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$

8) $\int x\sqrt{2-x} \, dx$

9) $\int \frac{x^3}{\sqrt{4-x^4}}dx$

10) $\int \frac{\left(\sqrt{x}+1\right)^4}{\sqrt{x}}dx$

11) $\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}dx$

12) $\int \frac{1}{\sqrt{1+\sqrt{x}}}dx$

Let $1+\sqrt{x}=u$

$\Rightarrow$

13) $\int \frac{3}{\sqrt[4]{x}+x}dx$

Let $1+x^{3/4}=u$

$\Rightarrow \frac{3}{4}x^{-1/4}dx = du$

$\Rightarrow dx = \frac{4}{3}\sqrt[4]{x}\,du$

$= \int \frac{1}{\sqrt[4]{x}(1+x^{3/4})}\,dx = \int \frac{1}{\sqrt[4]{x}\ u}\left(\frac{4}{3}\sqrt[4]{x}\,du\right)$

$= \frac{4}{3}\int \frac{1}{u}\,du = \frac{4}{3}\ln u + c$

$= \frac{4}{3}\ln\left(1+x^{3/4}\right)+c$

14) $\int \frac{1}{\sqrt{x}(1+x)}dx$

Let $\sqrt{x}=u \Rightarrow x=u^2$

$\Rightarrow dx = 2u\,du$

$= \int \frac{1}{u(1+u^2)}\,2u\,du = 2\int \frac{1}{1+u^2}\,du$

$= 2\tan^{-1}u + c \ = 2\tan^{-1}(\sqrt{x})+c$

Challenge

15) $\int \frac{3\sqrt{x}}{1+x^3}\,dx$

Let $x^{3/2} = u \Rightarrow x^3 = u^2$

$\Rightarrow \frac{3}{2}x^{1/2}dx = du$

$\Rightarrow dx = \frac{2}{3\sqrt{x}}\,du$

16) $\int x\, e^{x^2}\,dx$

17) $\int \left(\frac{e^{\sqrt{x}}}{\sqrt{x}}\right)dx$

18) $\int \left(\frac{\sqrt{\ln x}}{x}\right)dx$

19) $\int \left(\frac{\ln\sqrt{x}}{x}\right)dx$

20) $\int e^x\sqrt{e^x+4}\,dx$

21) $\int \left(\frac{\cos(1/x)}{x^2}\right)dx$

22) $\int \frac{x}{\sqrt{1-x^4}}\,dx$ Let $u = x^2$

$\Rightarrow$

23) $\int \frac{1+x}{1-x^2}\,dx$ Factor the denominator

24) $\int \frac{1+x}{1+x^2}\,dx$

25) $\int \sin^3 x \cos x\,dx$

26) $\int \frac{e^{\tan x}}{1-\sin^2 x}\,dx$

27) $\int x^2 \csc^2 x^3\,dx$

28) $\int \tan 4x\,dx$

29) $\int \frac{\ln(\sin x)}{\tan x}\, dx$

30) $\int \frac{\cos(\ln x)}{x}\, dx$

31) $\int \sec^2 x \sqrt{1 - 2\tan x}\, dx$

32) $\int \frac{x-1}{1+2x-x^2}\, dx$

33) $\int \frac{x^2}{\sqrt{1-x^6}}\, dx$

Let $u = x^3$

$\Rightarrow$

34) $\int \frac{3x^2}{1+x^6}\, dx$

35) $\int \frac{x-2}{x+7}\, dx$

36) $\int \cos(\tan 3x)\sec^2 3x \; dx$

37) $\int \frac{x\sqrt{x}}{1+x^5} \; dx$

Challenge

Let $x^{5/2} = u$

$\Rightarrow$

38) $\int \sec^2 x\sqrt{\tan x} \; dx$

39) $\int \frac{(1+\sin x)^5}{\sec x} \; dx$

40) $\int \frac{3}{(1+x^2)\tan^{-1} x} \; dx$

41) $\int \frac{\sqrt{1-\sin^2 x}}{1+\sin^2 x} \; dx$

Challenge

42) $\int \frac{2}{x^{\frac{2}{3}} - x^{\frac{5}{6}}} \; dx$

Let $u = x^{1/6}$

$\Rightarrow$

Evaluate each definite integral:

43) $\int_1^2 x^3\sqrt{x^4+1}\,dx$

44) $\int_1^e \frac{\ln x}{x}\,dx$

46) $\int_{-1}^1 \frac{t}{(1+t^2)^2}\,dt$

45) $\int_1^e \frac{1}{x\ln x + x}\,dx$

48) $\int_0^{\ln 2} \frac{e^t}{1+e^t}\,dt$

47) $\int_0^{\ln 2} \frac{e^t}{1+e^{2t}}\,dt$

49) If $\int_0^1 f(x)\, dx = 3$ Find $\int_0^{\frac{\pi}{2}} \cos x\, f(\sin x)\, dx$

50) If $\int_1^2 f(x)\, dx = 4$ Find $\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}}\, dx$

51) If $\int_1^2 f(x)\, dx = 3$ Find **A)** $\int_2^4 f\left(\frac{x}{2}\right)\, dx$

B) $\int_0^{\ln 2} e^x f(e^x)\, dx$

52) **A)** For the integral $I=\int_0^{10}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$, use a substitution to show that $I=\int_0^{10}\frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$ Use these two representations of I to evaluate I

Let $u=10-x \Rightarrow du=dx$

When $x=0 \Rightarrow u=10$ and $x=10 \Rightarrow u=0$

$$\Rightarrow \int_0^{10}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx=\int_0^{10}\frac{\sqrt{10-u}}{\sqrt{10-u}+\sqrt{u}}\,du$$

The integration value does not change when replace variable with another variable

$$I=\int_0^{10}\frac{\sqrt{10-x}}{\sqrt{10-x}+\sqrt{x}}\,dx \qquad \rightarrow (1)$$

Frome the question

$$I=\int_0^{10}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx \qquad \rightarrow (2)$$

Add (1) to (2)

$$I+I=\int_0^{10}\frac{\sqrt{10-x}}{\sqrt{10-x}+\sqrt{x}}\,dx+\int_0^{10}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$$

$$2I=\int_0^{10}\frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx=\int_0^{10}1\,dx$$

$$2I=x|_0^{10}=10-0$$

$$2I=10 \Rightarrow I=5$$

$$\Rightarrow I=\int_0^{10}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx=5$$

B) Generalize to $I=\int_0^{a}\frac{f(x)}{f(x)+f(a-x)}\,dx$ for any positive, then find the value of $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx$ and $\int_0^{5}\frac{f(x)}{f(x)+f(5-x)}\,dx$

53) When a patient is undergoing surgery, he is injected with anesthesia, and after t hours the concentration of anesthetic in the patient's blood is

$$C(t) = \frac{2t}{\sqrt{(36+t^2)^3}} \quad mg/cm^2$$

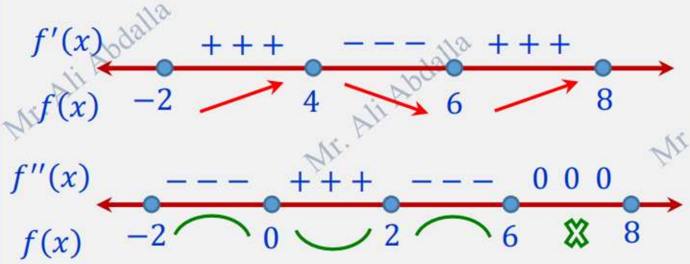
Find the average concentration of anesthesia in the blood during the first eight hours after injection

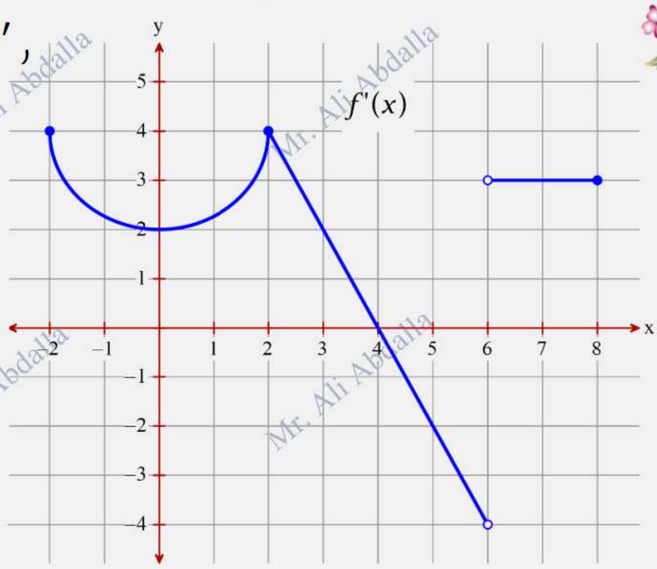
54) The Weather station observed the temperature C in a city after midnight, so it was found that it can be modeled with as the following:

$$T(t) = 3 - \frac{1}{3}(t-5)^2 \quad ℃$$

where t is the time after midnight. Find the average temperature in the city from 10 AM to 3 PM

55) The function f is continuous for all real values of x.
A portion of the graph of the function f', the derivative of f, on $[-2, 8]$.
The graph of f' is shown in the figure is shown on the right and consists of a semicircle and two linear pieces.





A) Find the x coordinate of each critical point of f on the interval $[-2, 8]$. Classify each critical point as a local maximum, a local minimum, or neither for f. Justify your answers.

Relative maximum at $x = 4$ since f' changes from $(+)$ to $(-)$

Relative minimum at $x = 6$ since f' changes from $(-)$ to $(+)$

B) Find the x coordinate of each point of inflection for the graph off on the open interval $(-2, 6)$. Justify your answer.

Point of inflection at $x = 0$ since f' changes from decreasing to increasing.

Point of inflection at $x = 2$ since f' changes from increasing to decreasing.

C) Find $\lim_{x\to 4} \frac{\int_2^x f'(t)dt - 4}{3(x-4)^2}$. Show the work that leads to your answer.

By direct substitution, Numerator is $\lim_{x\to 4} \int_2^x f'(t)dt - 4 = \underbrace{\int_2^4 f'(t)dt}_{\text{Area under curve of } f' \text{ from 2 to 4}} - 4 = 4 - 4 = 0$

Denominator: $\lim_{x\to 4} 3(x-4)^2 = 3(4-4)^2 = 0$

By using l'hopital twice:

$$\lim_{x\to 4} \frac{\int_2^x f'(t)dt - 4}{3(x-4)^2} = \lim_{x\to 4} \frac{f'(x) - 0}{6(x-4)^1(1)} \overset{\text{Direct subs.}}{\left(\frac{0}{0}\right)} = \lim_{x\to 4} \frac{f'(x)}{6x - 24} = \lim_{x\to 4} \frac{f''(x)}{6} = \frac{f''(4)}{6} = -\frac{2}{6} = -\frac{1}{3}$$

$f''(4)$ from the graph of f' slope at $x = 4$

D) Evaluate: $A) \int_{-2}^{1} f'(4-2x)dx$ $\qquad B) \int_{-2}^{3} f'(4-2x)dx$ B) Try it by

Let $u = 4 - 2x \Rightarrow du = -2dx \Rightarrow dx = -\frac{1}{2}du$ when $x = -2 \Rightarrow u = 8$ and when $x = 1 \Rightarrow u = 2$

$$\int_{-2}^{1} f'(4-2x)dx = -\frac{1}{2}\int_8^2 f'(u)\,du = \frac{1}{2}\int_2^8 f'(u)\,du = \frac{1}{2}\boxed{\int_2^8 f'(x)\,dx}$$

Area under curve of f' from 2 to 8

$$= \frac{1}{2}\left(\frac{1}{2}(2)(4) - \frac{1}{2}(2)(4) + (2)(3)\right) = 3$$

Fundamental theorem of calculus Part I

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

E) Let $g(x) = f'(x)\cdot x^2$ find $g'(3)$

$g'(x) = f''(x)\cdot x^2 + f'(x)\cdot 2x \Rightarrow g'(3) = f''(3)\cdot 3^2 + f'(3)\cdot 2(3)$

$\Rightarrow g'(3) = 9\,f''(3) + 6\,f'(3) = 9(-2) + 6(2) = -6 \Rightarrow \boxed{g'(3) = -6}$

$f''(3)$ from the graph of f' slope at $x = 3$ $\qquad$ $f'(3)$ from the graph of f' $f'(3) = 2$

Shortcuts: Integrals of Expressions Involving $(ax + b)$

Rule

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (\text{if } n \neq -1)$$

$$\int (3x-1)^2 \, dx = \frac{(3x-1)^3}{3(3)} + C = \frac{(3x-1)^3}{9} + C$$

$$\int (ax+b)^{-1} dx = \frac{1}{a} \ln|ax+b| + C$$

$$\int (3-2x)^{-1} dx = \frac{1}{(-2)} \ln|3-2x| + C = -\frac{1}{2} \ln|3-2x| + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$\int e^{-x+4} dx = \frac{1}{(-1)} e^{-x+4} + C = -e^{-x+4} + C$$

$$\int c^{ax+b} \, dx = \frac{1}{a \ln c} c^{ax+b} + C$$

$$\int 2^{-3x+4} \, dx = \frac{1}{(-3 \ln 2)} 2^{-3x+4} + C = -\frac{1}{3 \ln 2} 2^{-3x+4} + C$$

- $\displaystyle\int \sqrt{ax+b} \, dx = \frac{2}{3a}(ax+b)^{3/2} + C$

Review Exercises

Find the antiderivative.

1) $\int 4x \sec x^2 \tan x^2 \, dx$

2) $\int \tan x \, dx$

3) $\int \sqrt{3x+1} \, dx$

4) $\int e^x (1 - e^{-x}) \, dx$

5) $\int e^x (1 + e^x)^2 \, dx$

6) $\int 6x^2 \cos x^3 \, dx$

7) Find a function $f(x)$ satisfying $f(x) = e^{-2x}$ and $f(0) = 3$.

8) Determine the position function if the velocity is $v(t) = -32t + 10$ and the initial position is $s(0) = 2$.

9) Determine the position function if the acceleration is $a(t) = 6$ with initial velocity $v(0) = 10$ and initial position $s(0) = 0$.

10) Write out all terms and compute $\sum_{i=1}^{6}(i^2 + 3i)$.

11) Use summation rules to compute the sum of $\sum_{i=1}^{100}(i^2 + 2i)$.

12) Translate into summation notation and compute: the sum of the squares of the first 12 positive integers.

13) Compute the sum $\frac{1}{n^3}\sum_{i=1}^{n}(i^2 - i)$ and the limit of the sum as n approaches ∞.

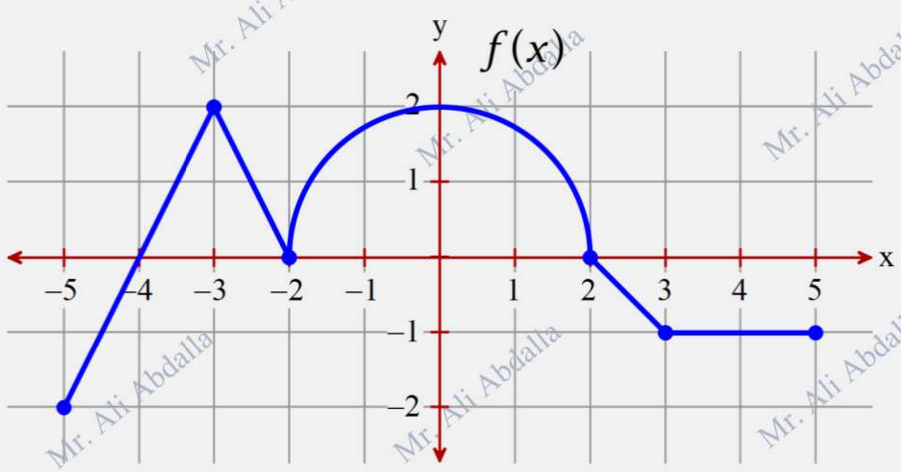
14) Use the velocity function to compute the distance traveled in the given time interval. $v(t) = 20e^{-t/2}$, $[0,2]$

15) Find the derivative of:

A) $f(x) = \int_{2}^{x} (\sin t^2 - 2)\, dt$

B) $f(x) = \int_{0}^{x^2} \sqrt{t^2 + 1}\, dt$

16)



The graph of $y = f(x)$ consists of four-line segments and a semicircle as shown in the figure above. Evaluate each definite integral by using geometric formulas.

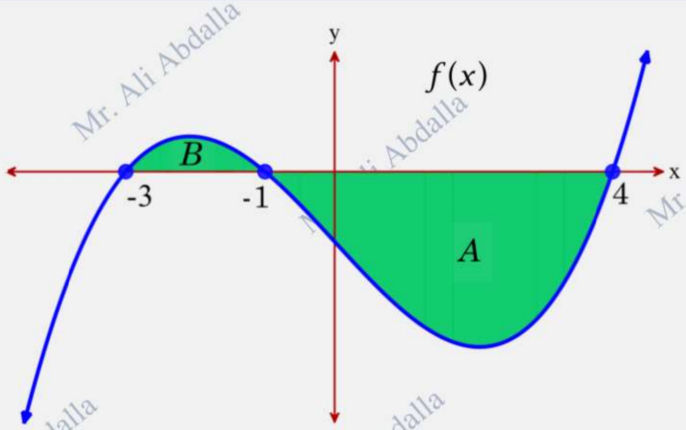
A) $\int_{-5}^{-2} f(x)\, dx$

B) $\int_{-2}^{2} f(x)\, dx$

C) $\int_{2}^{5} f(x)\, dx$

D) $\int_{-5}^{5} |f(x)|\, dx$

E) $\int_{-5}^{5} f(|x|)\, dx$



17) The graph of $y = f(x)$ is shown in the figure above. If A and B are positive numbers that represent the areas of the shaded regions, what is the value of $\int_{-3}^{4} f(x)\,dx - 2\int_{-1}^{4} f(x)\,dx$ in terms of A and B?

A) $-A - B$ B) $A + B$ C) $A - 2B$ D) $A - B$

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18) If f is the antiderivative of $\frac{\sqrt{x}}{1+x^3}$ such that $f(1) = 2$, then $f(3) =$

A) 1.845 B) 2.397 C) 2.906 D) 3.234

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19) If $f'(x) = \cos(2x - 1)$ such that $f(1) = 2$, then $f(5) =$

A) 1.825 B) 1.338 C) 1.285 D) 5.482

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20) If $\int_{-1}^{3} f(x + k)\,dx = 8$ where k is constant, then $\int_{k-1}^{k+3} f(x)\,dx =$

A) $8 - k$ B) $8 + k$ C) 8 D) $k - 8$

21) If f is continuous and $\int_1^8 f(x)\,dx = 15$, find the value of $\int_1^2 x^2 f(x^3)\,dx$.

22) $\int_1^e \frac{\cos(\ln x)}{x}\,dx =$

(A) $\frac{1}{\sin 1}$ (B) $\frac{1}{\cos 1}$ (C) $\sin(e)$ (D) $\sin 1$

23) Which of the following limits is equal to $\int_1^3 x^3\,dx$?

(A) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{i}{n})^3 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{i}{n})^3 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{2i}{n})^3 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{2i}{n})^3 \frac{2}{n}$

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