هيكل امتحانات نهاية الفصل 3 EoT3 Exam Coverage

Exam Duration مدة الامتحان	Maximum Overall Grade العلامة القصوى الممكنة	Marks per Question درجة كل سؤال	Type of Questions طبيعة الاسئلة	Number of Question عدد الاسئلة	Grade الصف	Subject المادة
120	100	5	MCQs اختيار من متعدد	25	12 Advanced	Math رياضيات

Q	Learning Outcome***	Example/Exercise	Page
1	**Find the area between two curves using definite integrations**	**(5-10)**	**414**
2	**Compute the area of a region using definite integration with y as a variable**	**(19,20,22,24)**	**414**
3	**Compute volume by means of definite integration using areas of cross sections**	**(1-4) Not writing exercises**	**429**
4	**Find the volume of a solid of revolution using the method of disks**	**(17a,19a,25a+b) (27b+c,28a)**	**430-431**
5	**Find the volume of a solid of revolution by using the method of washers**	**(17b,18,19b,20,25c+d+e+f) (27a+d+e+f,28b+c+d)**	**430-431**
6	**Find the volume of a solid of revolution by using the method of cylindrical shells**	**(1,2) Not writing exercises-- (3-8)**	**438-439**
7	**Find arc length in a given interval using definite integration**	**(5-10)**	**446**
8	**Find surface area of a solid of revolution using definite integration**	**(29-36)**	**447**
9	**Solve mathematical problems involving applications on arc length or surface area**	**(23,24)**	**447**
10	**Solve physical problems involving velocity**	**(1-4)**	**455**
11	**Solve problems on projectiles**	**(17-23)**	**456**
12	**Compute integrals using direct computation and rules**	**(3-10)**	**489**
13	**Compute various interlays using integration by substitution**	**(14,16,19-24)**	**489**
14	**Compute integrals using completing a square before integrating**	**(11-13)**	**489**
15	**Use integration by parts to compute definite and indefinite integrals**	**(1-6,23,24)**	**496**
16	**Use integration by parts to compute definite and indefinite integrals**	**(19-22)**	**496**
17	**Integrate functions of the form sin^n(x) cos^m(x)**	**(1-8)**	**507**
18	**Integrate functions of the form sec^m(x) tan^n(x)**	**(9-16)**	**507**
19	**Integrate trigonometric functions using the substitution x=a sin(y)**	**(21-26)**	**507**
20	**Integrate trigonometric functions using the substitution x=a tan(y)**	**(33-41)**	**507**
21	**Integrate trigonometric functions using the substitution x=a sec(y)**	**(27-32)**	**507**
22	**Integrate rational functions using partial fractions in different cases**	**(1-12)**	**516**
23	**Learn differential equations of the form y'=ky and their general solution**	**(1-8)**	**533**
24	**Solve problems involving differential equations of the form y'=ky satisfying an indicated initial condition**	**(28-30) (31,32)**	**534-535**
25	**Find the general solution of separable differential equations of first order**	**(5-16)**	**544**

Best 20 answers out of 25 will count. Example: 14 correct answers yield a grade of 70/100, while 20 and 23 correct answers yield a (full) grade of 100/100 each.	**تحتسب أفضل 20 إجابة من 25. مثال 14 إجابة صحيحة تعطي علامة 70/100 بينما 20 أو 23 إجابة صحيحة تعطي العلامة الكاملة أي100/100**
Questions might appear in a different order in the actual exam.	**قد تظهر الأسئلة بترتيب مختلف في الامتحان الفعلي**
As it appears in the textbook/LMS/SoW.	**كما وردت في كتاب الطالب و الخطة الفصلية**

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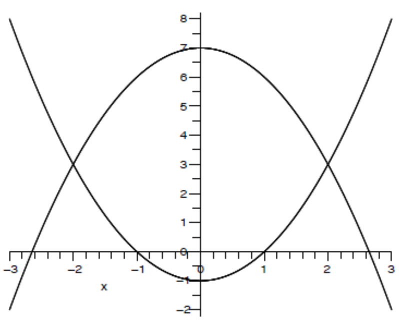
Q	Learning Outcome***	Example/Exercise	Page
1	**Find the area between two curves using definite integrations**	**(5-10)**	**414**
2	**Compute the area of a region using definite integration with y as a variable**	**(19,20,22,24)**	**414**

In exercises 5–12, sketch and find the area of the region deter- mined by the intersections of the curves

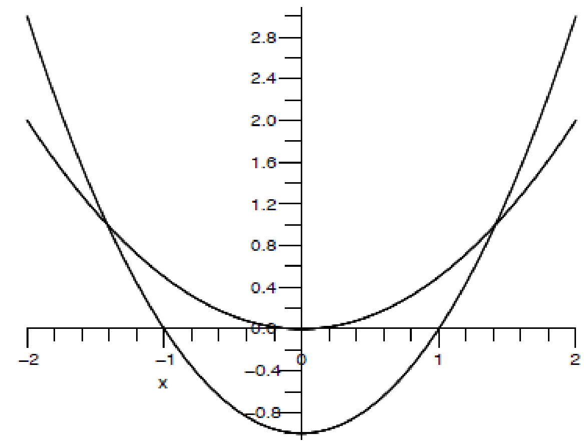
5. $y = x^2 - 1, y = 7 - x^2$

$$\text{Area} = \int_{-2}^{2} [7 - x^2 - (x^2 - 1)]\,dx$$
$$= \left(8x - \frac{2x^3}{3}\right)\Big|_{-2}^{2}$$
$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$$
$$= \frac{64}{3}$$



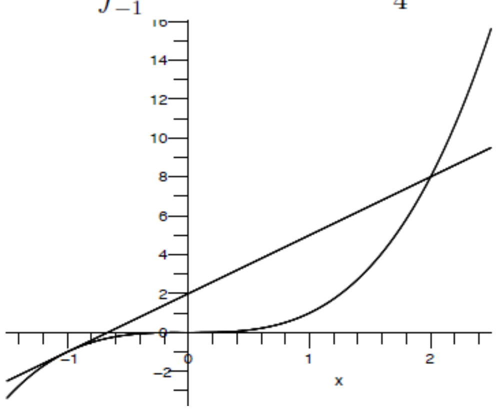
6. $y = x^2 - 1, y = \frac{1}{2}x^2$

$$\text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x^2}{2} - (x^2 - 1)\right] dx$$
$$= \frac{4\sqrt{2}}{3}$$



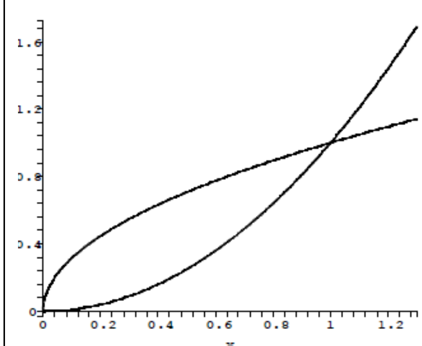
7. $y = x^3, y = 3x + 2$

$$\text{Area} = \int_{-1}^{2} (3x + 2 - x^3)\,dx = \frac{27}{4}$$



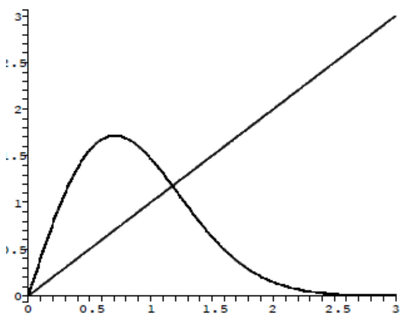
8. $y = \sqrt{x}, y = x^2$

$$\text{Area} = \int_{0}^{1} (\sqrt{x} - x^2)\,dx = \frac{1}{3}$$



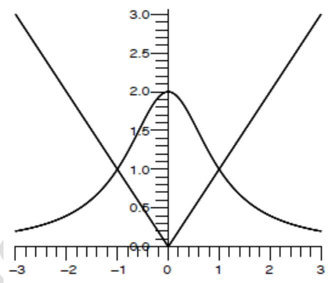
9. $y = 4xe^{-x^2}, y = |x|$

$$\text{Area} = \int_0^{\sqrt{\ln 4}} \left(4xe^{-x^2} - x\right) dx$$
$$= -2e^{-x^2} - \frac{x^2}{2}\Big|_0^{\sqrt{\ln 4}}$$
$$= -2\left[\frac{1}{4} - 1\right] - \frac{\ln 4}{2}$$
$$= \frac{3 - \ln 4}{2}$$



10. $y = \dfrac{2}{x^2+1}, y = |x|$

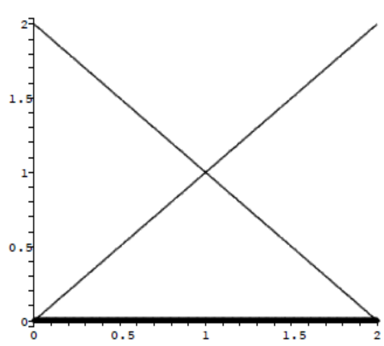
$$\text{Area} = \int_{-1}^{0} \left(\frac{2}{x^2+1} + x\right) dx$$
$$+ \int_0^1 \left(\frac{2}{x^2+1} - x\right) dx$$
$$= (2\tan^{-1} x + \frac{x^2}{2})\Big|_{-1}^{0}$$
$$+ (2\tan^{-1} x - \frac{x^2}{2})\Big|_0^1$$
$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) + \left(\frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{\pi}{2} - 1$$



In exercises 19–26, sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral. Verify your answers to exercises 19–21 with a basic geometric area formula.

19. $y = x, y = 2 - x, y = 0$

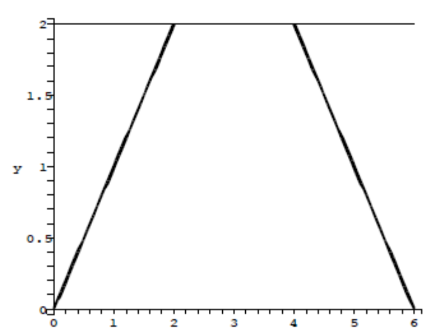
$$\text{Area} = \int_0^1 [(2-y) - y]dy$$
$$= \int_0^1 [2 - 2y]dy$$
$$= (2y - y^2)\big|_0^1$$
$$= 1 - 0 = 1$$



Area of triangle $= \frac{1}{2}(base)(height)$
$= \frac{1}{2} \cdot (2) \cdot (1) = 1$

20. $y = x, y = 2, y = 6 - x, y = 0$

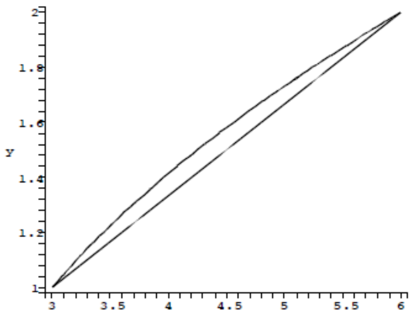
$$\text{Area} = \int_0^2 [(6-y) - y]dy$$
$$= \int_0^2 (6 - 2y)dy$$
$$= (6y - y^2)\big|_0^2$$
$$= (12 - 4) - (0 - 0)$$
$$= 8$$



Area of Trapazium $= \frac{1}{2}(a + b)(h)$
$= \frac{1}{2} \cdot (8) \cdot (2) = 8$

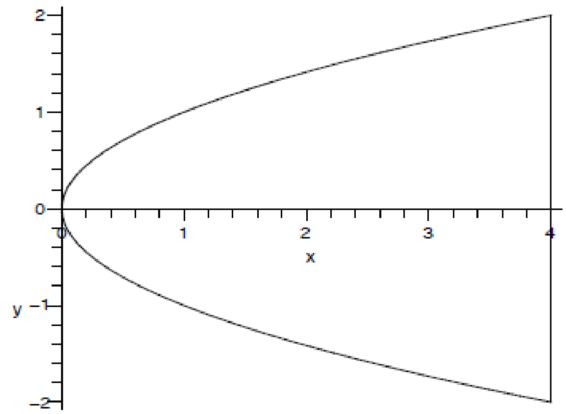
22. $x = 3y, x = 2 + y^2$

$$\text{Area} = \int_1^2 [3y - (2 + y^2)]\,dy$$
$$= \left(\frac{3}{2}y^2 - 2y - \frac{y^3}{3}\right)\Bigg|_1^2$$
$$= \left(6 - 4 - \frac{8}{3}\right) - \left(\frac{3}{2} - 2 - \frac{1}{3}\right)$$
$$= \frac{1}{6}$$



24. $x = y^2, x = 4$

$$\text{Area} = \int_{-2}^{2} (4 - y^2)\,dy = \frac{32}{3}$$



Q	Learning Outcome***	Example/Exercise	Page
3	Compute volume by means of definite integration using areas of cross sections	(1-4) Not writing exercises	429
4	Find the volume of a solid of revolution using the method of disks	(17a,19a,25a+b) (27b+c,28a)	430-431
5	Find the volume of a solid of revolution by using the method of washers	(17b,18,19b,20,25c+d+e+f) (27a+d+e+f,28b+c+d)	430-431

In exercises 1–4, find the volume of the solid with cross- sectional area *A*(*x*).

1. $A(x) = x + 2, -1 \le x \le 3$

$$V = \int_{-1}^{3} A(x)\,dx = \int_{-1}^{3} (x+2)\,dx$$
$$= \left(\frac{x^2}{2} + 2x\right)\Bigg|_{-1}^{3} = \left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)$$
$$= 12$$

2. $A(x) = 10e^{0.01x}, 0 \le x \le 10$

$$V = \int_0^{10} 10e^{0.01x}\,dx = \left(1000e^{0.01x}\right)\Big|_0^{10}$$
$$= 1000(e^{0.1} - 1)$$

3. $A(x) = \pi(4 - x)^2, 0 \le x \le 2$

$$V = \pi \int_0^2 (4 - x)^2\,dx = -\frac{\pi}{3}(4 - x)^3\Big|_0^2$$
$$= -\frac{\pi}{3}(8 - 64) = \frac{56\pi}{3}$$

4. $A(x) = 2(x + 1)^2, 1 \le x \le 4$

$$V = \int_1^4 2(x+1)^2\,dx$$
$$= \int_1^4 (2x^2 + 4x + 2)\,dx = 78$$

	Axis of revolution			
Horizontal	x-axis y=0		$V=\pi\int_a^b (R(x))^2-(r(x))^2dx$	Xaxis Y=0
Horizontal	Y=k	+	$V=\pi\int_a^b (k-R(x))^2-(k-r(x))^2dx$	Y=6
Horizontal	Y=k	-	$V=\pi\int_a^b (R(x)-k)^2-(r(x)-k)^2dx$	y=-2
Vertical	y-axis X=0		$V=\pi\int_c^d (R(y))^2-(r(y))^2dy$	yaxis X=0
Vertical	X=L	+	$V=\pi\int_c^d (L-R(y))^2-(L-r(y))^2dy$	X=2
Vertical	X=L	-	$V=\pi\int_c^d (R(y)-L)^2-(r(y)-L)^2dy$	X=-4

In exercises, 17–20, compute the volume of the solid formed by revolving the given region about the given line

17. Region bounded by $y = 2 - x$, $y = 0$ and $x = 0$ about (a) the x-axis; (b) $y = 3$

(a) the x-axis;

$$\text{(a)}\ V = \pi\int_0^2 (2-x)^2dx$$
$$= -\pi\left(\frac{(2-x)^3}{3}\right)\Bigg|_0^2$$
$$= \frac{8\pi}{3}$$

(b) $y = 3$

$$\text{(b)}\ V = \pi\int_0^2 \left[3^2 - \{3-(2-x)\}^2\right]dx$$
$$= \pi\int_0^2 \left[9 - \{1+x\}^2\right]dx$$
$$= \pi\left[9x\big|_0^2 - \frac{(1+x)^3}{3}\Bigg|_0^2\right]$$
$$= \pi\left[18 - \frac{3^3-1^3}{3}\right] = \frac{28\pi}{3}$$

18. Region bounded by $y = x^2$, $y = 4 - x^2$ about (a) the x-axis; (b) $y = 4$

(a) the x-axis;

$$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left[(4-x^2)^2 - (x^2)^2\right]dx$$
$$= \pi\left[16x - \frac{8x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$$
$$= \pi\left(\frac{64\sqrt{2}}{3}\right)$$

(b) $y = 4$

$$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}} (4-x^2)^2 - (x^2)^2dx$$
$$= \pi\left(\frac{64\sqrt{2}}{3}\right)$$

19. Region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 4$

(a) the y-axis,

$$\text{(a)}\quad V = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy$$

$$= \pi \left(\frac{y^5}{5}\right)\Big|_0^2 = \frac{32\pi}{5}$$

(b) $x = 4$

$$\text{(b)}\quad V = \pi \int_0^2 (4)^2 dy - \pi \int_0^2 (4 - y^2)^2 dy$$

$$= \pi \int_0^2 (-y^4 + 8y^2) dy$$

$$= \pi \left(-\frac{y^5}{5} + \frac{8y^3}{3}\right)\Big|_0^2$$

$$= \pi \left[\left(-\frac{32}{5} + \frac{64}{3}\right) - (0 + 0)\right]$$

$$= \frac{224\pi}{15}$$

20. Region bounded by $y = x^2$ and $x = y^2$ about (a) the y-axis; (b) $x = 1$

(a) the y-axis,

$$V = \pi \int_0^1 (\sqrt{y})^2 dy - \pi \int_0^1 (y^2)^2 dy$$

$$= \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= \frac{3\pi}{10}$$

(b) $x = 1$

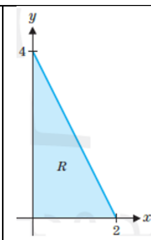
$$\text{(b)}\quad V = \pi \int_0^1 (1 - y^2)^2 dy - \pi \int_0^1 (1 - \sqrt{y})^2 dy$$

$$= \pi \int_0^1 (y^4 - 2y^2 - y + 2\sqrt{y})\, dy$$

$$= \pi \left(\frac{y^5}{5} - \frac{2y^3}{3} - \frac{y^2}{2} + \frac{4y^{\frac{3}{2}}}{3}\right)\Big|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right) = \frac{11\pi}{30}$$

25. Let R be the region bounded by y = 4 – 2x, the x-axis and the y-axis. Compute the volume of the solid formed by revolving R about the given line.



(a) the y-axis

$$\text{(a)}\quad V = \int_0^4 \pi \left(\frac{4 - y}{2}\right)^2 dy$$

$$= \frac{\pi}{4} \int_0^4 (16 - 8y + y^2) dy$$

$$= \frac{\pi}{4} \left[16y - 4y^2 + \frac{y^3}{3}\right]_0^4$$

$$= \frac{\pi}{4} \left[64 - 64 + \frac{64}{3}\right] = \frac{16\pi}{3}$$

(b) the x-axis

$$\text{(b)}\quad V = \int_0^2 \pi (4 - 2x)^2 dx$$

$$= \pi \int_0^2 (16 - 16x + 4x^2) dx$$

$$= \pi \left[16x - 16\frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2$$

$$= \pi \left[32 - 32 + \frac{32}{3}\right] = \frac{32\pi}{3}$$

(c) $y = 4$

$$V = \int_0^2 \pi(4)^2 dx - \int_0^2 \pi(2x)^2 dx$$
$$= \pi\int_0^2 (16 - 4x^2)dx$$
$$= \pi\left[16x - \frac{4x^3}{3}\right]_0^2$$
$$= \pi\left[32 - \frac{32}{3}\right] = \frac{64\pi}{3}$$

(d) $y = -4$

$$\text{(d) } V = \int_0^2 \pi(8-2x)^2 dx - \int_0^2 \pi(4)^2 dx$$
$$= \pi\int_0^2 (64 - 32x + 4x^2 - 16)dx$$
$$= \pi\left[48x - 32\frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2$$
$$= \pi\left[96 - 64 + \frac{32}{3}\right] = \frac{128\pi}{3}$$

(e) $x = 2$

$$\text{(e) } V = \int_0^4 \pi(2)^2 dy - \int_0^4 \pi\left(\frac{y}{2}\right)^2 dy$$
$$= \pi\int_0^4 \left(4 - \frac{y^2}{4}\right)dy$$
$$= \pi\left[4y - \frac{1}{4}\cdot\frac{y^3}{3}\right]_0^4$$
$$= \pi\left[16 - \frac{16}{3}\right] = \frac{32\pi}{3}$$

(f) $x = -2$

$$\text{(f) } V = \int_0^4 \pi\left(\frac{8-y}{2}\right)^2 dy - \int_0^4 \pi(2)^2 dy$$
$$= \pi\int_0^4 \left(\frac{64 - 16y + y^2}{4} - 4\right)dy$$
$$= \frac{\pi}{4}\left[64y - 16\frac{y^2}{2} + \frac{y^3}{3} - 16y\right]_0^4$$
$$= \pi\left[64 + \frac{64}{3}\right] = \frac{256\pi}{3}$$

27.Let R be the region bounded by $y = x^2$, $y = 0$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

a-the y-axis

$$V = \int_0^1 \pi(1)^2 dy - \int_0^1 \pi(\sqrt{y})^2 dy$$
$$= \pi\int_0^1 (1 - y)dy$$
$$= \pi\left(y - \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{2}$$

b-the x-axis

$$\text{(b) } V = \int_0^1 \pi(x^2)^2 dx$$
$$= \pi\frac{x^5}{5}\Bigg|_0^1 = \frac{\pi}{5}$$

c- $x = 1$

$$\text{(c) } V = \int_0^1 \pi(1 - \sqrt{y})^2 dy$$
$$= \pi\int_0^1 \left(1 - 2y^{1/2} + y\right)dy$$
$$= \pi\left(y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{6}$$

d- $y = 1$

$$V = \int_0^1 \pi(1)^2 dx - \int_0^1 \pi(1 - x^2)^2 dx$$
$$= \pi\int_0^1 (2x^2 - x^4)\,dx$$
$$= \pi\left(\frac{2}{3}x^3 - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{7\pi}{15}$$

e- $x = -1$

$$V = \int_0^1 \pi(2)^2 dy - \int \pi(1 + \sqrt{y})^2 dy$$
$$= \pi\int_0^1 \left(3 - 2y^{1/2} - y\right)dy$$
$$= \pi\left(3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{7\pi}{6}$$

f- $y = -1$

$$V = \int_0^1 \pi(x^2 + 1)^2 dx$$
$$= -\int_0^1 \pi(1)^2 dx$$
$$= \pi\int_0^1 (x^4 + 2x^2)\,dx$$
$$= \pi\left(\frac{x^5}{5} + \frac{2}{3}x^3\right)\Bigg|_0^1 = \frac{13\pi}{15}$$

28.Let *R* be the region bounded by *y* = *x, y* = −*x* and *x* = 1. Compute the volume of the solid formed by revolving *R* about the given line

a- the *x*-axis

(a) $V = \int_0^1 \pi x^2 dx = \frac{\pi}{3}$

b- the *y*-axis

(b) $V = \int_{-1}^0 \pi\left[1-(1+y)^2\right] dy + \int_0^1 \pi\left[1-(1-y)^2\right] dy = \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$

c- *y* = 1

(c) $V = \int_0^1 \pi\left[(1+x)^2 - (1-x)^2\right] dx = 2\pi$

d- *y* = −1

(d) $V = \int_0^1 \pi\left[(1+x)^2 - (1-x)^2\right] dx = 2\pi$

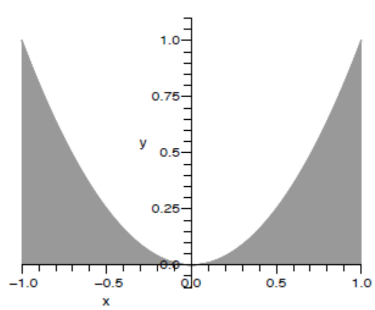
Q	Learning Outcome***	Example/Exercise	Page
6	**Find the volume of a solid of revolution by using the method of cylindrical shells**	**(1,2) Not writing exercises--(3-8)**	**438-439**

In exercises 1–8, sketch the region, draw in a typical shell, iden tify the radius and height of each shell and compute the volume.

1. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = 2$

1. Radius of a shell: $r = 2 - x$
Height of a shell: $h = x^2$

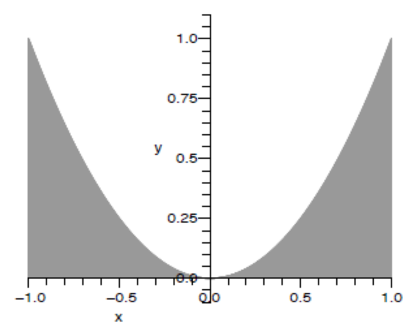
$$V = \int_{-1}^1 2\pi(2-x)x^2 dx = 2\pi\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_{-1}^1 = \frac{8\pi}{3}$$



2. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = -2$

2. Radius of a shell: $r = 2 + x$
Height of a shell: $h = x^2$

$$V = \int_{-1}^1 2\pi(2+x)x^2 dx = \frac{8\pi}{3}$$



$$V = \lim_{n\to\infty} \sum_{i=1}^n 2\pi c_i f(c_i)\,\Delta x = \int_a^b 2\pi \underbrace{x}_{\text{radius}} \underbrace{f(x)}_{\text{height}} \underbrace{dx.}_{\text{thickness}}$$

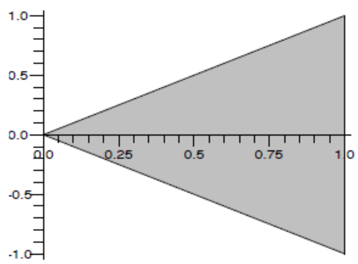
3. The region bounded by $y = x, y = -x$ and $x = 1$ revolved about the y-axis

3. Radius of a shell: $r = x$

Height of a shell: $h = 2x$

$$V = \int_0^1 2\pi x(2x)dx$$

$$= \frac{4\pi}{3}x^3\Big|_0^1 = \frac{4\pi}{3}$$

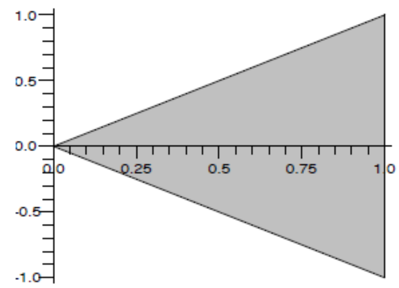


4. The region bounded by $y = x, y = -x$ and $x = 1$ revolved about $x = 1$

4. Radius of a shell: $r = 2 - x$.

Height of a shell: $h = 2x$.

$$V = \int_0^1 2\pi(2-x)(2x)dx = \frac{8\pi}{3}$$



5. The region bounded by $y = \sqrt{x^2+1}$ and $y = 0$, $0 \le x \le 4$ revolved about $x = 0$

5. Radius of a shell: $r = x$.

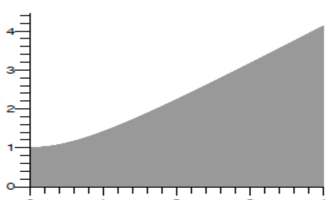
eight of a shell: $h = f(x) = \sqrt{x^2+1}$.

$$V = \int_0^4 2\pi x\sqrt{x^2+1}dx$$

$$= \pi\int_0^4 2x\sqrt{x^2+1}dx$$

$$= \pi\left(\frac{2(x^2+1)^{\frac{3}{2}}}{3}\right)\Bigg|_0^4 = \frac{2\pi}{3}\left[(17)^{\frac{3}{2}} - 1\right]$$

$$\approx 144.7076$$

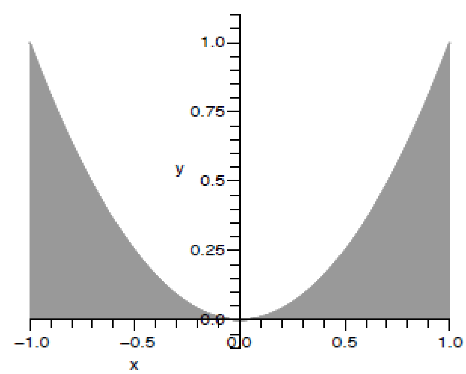


6. The region bounded by $y = x^2$ and $y = 0, -1 \le x \le 1$, revolved about $x = 2$

6. Radius of a shell: $r = 2 - x$.

Height of a shell: $h = f(x) = x^2$.

$$V = \int_{-1}^{1} 2\pi\,(2-x)\;x^2dx = \frac{8\pi}{3}$$



7. The region bounded by $x^2 + y^2 = 1$ revolved about $y = 2$

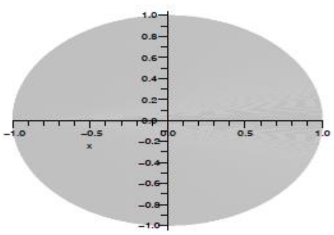
7. Radius of a shell: $r = 2 - y$.
Height of a shell: $h = f(y) = 2\sqrt{1-y^2}$.

$$V = \int_{-1}^{1} 2\pi\,(2-y)\,2\sqrt{1-y^2}\,dy$$

$$= 4\pi \int_{-1}^{1} (2-y)\sqrt{1-y^2}\,dy$$

$$= 8\pi \int_{-1}^{1} \sqrt{1-y^2}\,dy - 4\pi \int_{-1}^{1} y\sqrt{1-y^2}\,dy$$

$$= 16\pi\left(\frac{\pi}{4}\right) - 0 = 4\pi^2$$



8. The region bounded by $x^2 + y^2 = 2y$ revolved about $y = 4$

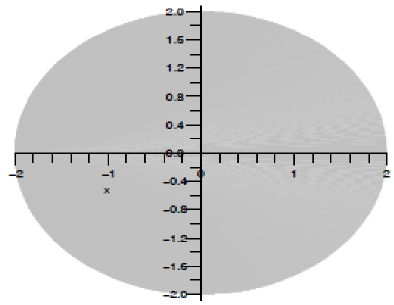
8. Radius of a shell: $r = 4 - y$.
Height of a shell: $h = f(y) = 2\sqrt{4-y^2}$.

$$V = \int_{-2}^{2} 2\pi\,(4-y)\,2\sqrt{4-y^2}\,dy$$

$$= 4\pi \int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

$$= 2\left(8\pi \int_{-2}^{2} \sqrt{4-y^2}\,dy - 2\pi \int_{-2}^{2} y\sqrt{4-y^2}\,dy\right)$$

$$= 2\,(8\pi\,(2\pi)) - 0 = 32\pi^2$$



Q	Learning Outcome***	Example/Exercise	Page
7	Find arc length in a given interval using definite integration	(5-10)	446
8	Find surface area of a solid of revolution using definite integration	(29-36)	447
9	Solve mathematical problems involving applications on arc length or surface area	(23,24)	447

arc length	$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$	surface area	$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx,$

In exercises 5–14, compute the arc length exactly.

5. $y = 2x + 1,\ 0 \le x \le 2$

5. This is a straight line segment from $(0, 1)$ to $(2, 5)$. As such, its length is

$$s = \sqrt{(5-1)^2 + (2-0)^2}$$

$$= \sqrt{20} = 2\sqrt{5}$$

6. $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$

6. $s = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\,dx$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= (\sin^{-1} x)\Big|_{-1}^{1} = \pi$$

7. $y = 4x^{3/2} + 1,\ 1 \le x \le 2$

7. $y'(x) = 6x^{1/2}$, the arc length integrand is $\sqrt{1+(y')^2} = \sqrt{1+36x}$.

Let $u = 1 + 36x$ then

$$s = \int_1^2 \sqrt{1+36x}\,dx$$
$$= \int_{37}^{73} \sqrt{u}\left(\frac{du}{36}\right)$$
$$= \frac{2}{3(36)} u^{3/2}\Big|_{37}^{73}$$
$$= \frac{1}{54}(73\sqrt{73} - 37\sqrt{37})$$
$$\approx 7.3824$$

8. $y = \frac{1}{4}(e^{2x} + e^{-2x}),\ 0 \le x \le 1$

8. $s = \int_0^1 \sqrt{1 + (e^{2x} - e^{-2x})^2}\,dx$

$$= \int_0^1 \sqrt{e^{4x} - 1 + e^{-4x}}\,dx$$
$$\approx 3.056$$

9. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x,\ 1 \le x \le 2$

9. $y'(x) = \frac{2x}{4} - \frac{1}{2x} = \frac{1}{2}\left(x - \frac{1}{x}\right)$

$$1 + (y')^2 = 1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)$$
$$= \frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)$$
$$= \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^2$$
$$s = \frac{1}{2}\int_1^2 \left(x + \frac{1}{x}\right)dx$$
$$= \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\Big|_1^2$$
$$= \frac{1}{2}\left(\frac{3}{2} + \ln 2\right)$$
$$\approx 1.0965$$

10. $y = \frac{1}{6}x^3 + \frac{1}{2x},\ 1 \le x \le 3$

10. $y'(x) = \frac{1}{2}(x^2 + x^{-2})$

$$s = \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,dx$$
$$= \frac{1}{2}\int_1^3 \frac{\sqrt{x^8 + 6x^4 + 1}}{x^2}dx$$
$$\approx 5.152$$

23. A rope is to be hung between two poles 40 meters apart. If the rope assumes the shape of the catenary $y = 10(e^{x/20} + e^{-x/20}),\ -20 \le x \le 20$, compute the length of the rope.

23. Here $f(x) = 10\left(e^{x/20} + e^{-x/20}\right)$

$$\Rightarrow f'(x) = \frac{10}{20}\left(e^{x/20} - e^{-x/20}\right)$$
$$1 + \left(f'(x)\right)^2 = 1 + \left(\frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)\right)^2$$
$$= \left(\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right)^2$$

Now,

$$s = \int_{-20}^{20} \frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)dx$$
$$= \int_0^{20} \left(e^{x/20} + e^{-x/20}\right)dx$$
$$= 20\left(e^{x/20} - e^{-x/20}\right)\Big|_0^{20}$$
$$= 20\left(e - e^{-1}\right) \approx 47.0080$$

24. A rope is to be hung between two poles 60 meters apart. If the rope assumes the shape of the catenary $y = 15(e^{x/30} + e^{-x/30})$, $-30 \le x \le 30$, compute the length of the rope.

24. $$s = \int_{-30}^{30} \sqrt{1 + \left[\frac{1}{2}\left(e^{x/30} - e^{-x/30}\right)\right]^2}\, dx$$
$$= \int_{-30}^{30} \frac{1}{2}\left(e^{x/30} + e^{-x/30}\right) dx$$
$$= \left(15e^{x/30} - 15e^{-x/30}\right)\Big|_{-30}^{30}$$
$$= 30e - 30e^{-1} \approx 70.51207161\, ft.$$

In exercises 29–36, set up the integral for the surface area of the surface of revolution and approximate the integral with a numerical method.

29. $y = x^2$, $0 \le x \le 1$, revolved about the x-axis

$$S = 2\pi \int_0^1 y\, ds$$
$$= 2\pi \int_0^1 x^2\sqrt{1 + (2x)^2}dx$$
$$\approx 3.8097$$

30. $y = \sin x$, $0 \le x \le \pi$, revolved about the x-axis

$$S = \int_0^{\pi} 2\pi \sin x\sqrt{1 + \cos^2 x}dx$$
$$\approx 14.42360$$

31. $y = 2x - x^2$, $0 \le x \le 2$, revolved about the x-axis

$$S = 2\pi \int_0^2 y\, ds$$
$$= 2\pi \int_0^2 (2x - x^2)\sqrt{1 + (2 - 2x)^2}dx$$
$$\approx 10.9654$$

32. $y = x^3 - 4x$, $-2 \le x \le 0$, revolved about the x-axis

$$S = \int_{-2}^{0} 2\pi(x^3 - 4x)\sqrt{1 + (3x^2 - 4)^2}dx$$
$$\approx 67.06557$$

33. $y = e^x$, $0 \le x \le 1$, revolved about the x-axis

$$S = 2\pi \int_0^1 y\, ds$$
$$= 2\pi \int_0^1 e^x\sqrt{1 + e^{2x}}dx \approx 22.9430$$

34. $y = \ln x$, $1 \le x \le 2$, revolved about the x-axis

$$S = \int_1^2 2\pi \ln x\sqrt{1 + \frac{1}{x^2}}dx$$
$$\approx 2.86563$$

35. $y = \cos x$, $0 \le x \le \pi/2$, revolved about the x-axis

$$S = 2\pi \int_0^{\pi/2} y\, ds$$
$$= 2\pi \int_0^{\pi/2} \cos x\sqrt{1 + \sin^2 x}dx$$
$$\approx 7.2117$$

36. $y = \sqrt{x}$, $1 \le x \le 2$, revolved about the x-axis

$$S = \int_1^2 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}dx \approx 8.28315$$

Q	Learning Outcome***	Example/Exercise	Page
10	Solve physical problems involving velocity	(1-4)	455
11	Solve problems on projectiles	(17-23)	456

In exercises 1–4, identify the initial conditions $y(0)$ and $y'(0)$.

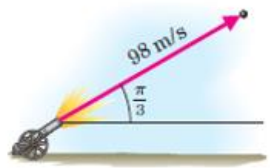
1. An object is dropped from a height of 80 ft.
$y(0) = 80, y'(0) = 0$

2. An object is dropped from a height of 100 ft.
$y(0) = 100, y'(0) = 0$

3. An object is released from a height of 60 ft with an upward velocity of 10 ft/s.
$y(0) = 60, y'(0) = 10$

4. An object is released from a height of 20 ft with a downward velocity of 4 ft/s.
$y(0) = 20, y'(0) = -4$

17. An object is launched at angle $\theta = \pi/3$ radians from the horizontal with an initial speed of 98 m/s. Determine the time of flight and the horizontal range. Compare to example 5.4.



The starting point is
$y'' = -9.8,\ y'(0) = 98\sin(\pi/3) = 49\sqrt{3}$.
We get $y(t) = -4.9t^2 + ty'(0)$
$= -4.9t(t - [v(0)/4.9])$
$= -4.9t(t - 10\sqrt{3})$
The flight time is $10\sqrt{3}$. As to the horizontal range, we have $x'(t)$ constant and forever equal to $98\cos(\pi/3) = 49$. Therefore $x(t) = 49t$ and in this case, the horizontal range is $49(10\sqrt{3})$ (meters).

18. Find the time of flight and horizontal range of an object launched at angle 30° with initial speed 40 m/s. Repeat with an angle of 60°.

Here $y'(0) = 40\sin\left(\frac{\pi}{6}\right) = 20$
Therefore $y(t) = -4.9t^2 + 20t$
$= t(-4.9t + 20)$
$\Rightarrow$ the time of flight $= t = \frac{20}{4.9} = 4.082$
Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\frac{\pi}{6}\right) = 20\sqrt{3}$

Therefore
$x(t) = 20\sqrt{3}t$ and
$x(4.082) = 20(1.7321)(4.082) = 141.3919$
Repeating the same for the angle 60^0
$y'(0) = 40\sin\left(\frac{\pi}{3}\right) = 34.6410$

Therefore
$y(t) = -4.9t^2 + (34.6410)t$
$\Rightarrow y(t) = t(-4.9t + 34.6410)$
$\Rightarrow$ the time of flight $= t = \frac{34.6410}{4.9} = 7.0696$
Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\frac{\pi}{3}\right) = 20$
Therefore $x(t) = 20t$ and
$x(7.0696) = 20(7.0696) = 141.3919$

19. Repeat example 5.5 with an initial angle of 6°. By trial and error, find the smallest and largest angles for which the serve will be in.

This problem modifies Example 5.5 by using a service angle of 6° (where the Example 5.5 used 7°) and no other changes. Here the serve hits the net.

Next we want to find the range for which the serve will be in.

If θ is the angle, then the initial conditions are
$x'(0) = 176\cos\theta$, $x(0) = 0$
$y'(0) = 176\sin\theta$, $y(0) = 10$

Integrating $x''(t) = 0$ and $y''(t) = -32$, then using the initial conditions gives
$x'(t) = 176\cos\theta$
$x(t) = 176(\cos\theta)t$
$y'(t) = -32t + 176\sin\theta$
$y(t) = -16t^2 + 176(\sin\theta)t + 10$

To make sure the serve is in, we see what happens at the net and then when the ball hits the ground. First, the ball passes the net when $x = 39$ or when $39 = 176(\cos\theta)t$. Solving gives $t = \dfrac{39}{176\cos\theta}$ Plugging this in for the function $y(t)$ gives

$$y\left(\frac{39}{176\cos\theta}\right)$$
$$= -16\left(\frac{39}{176\cos\theta}\right)^2 + 176(\sin\theta)\left(\frac{39}{176\cos\theta}\right) + 10$$
$$= -\frac{1521}{1936}\sec^2\theta + 39\tan\theta + 10$$

We want to ensure that this value is greater than 3 so we determine the values of θ that give $y > 3$ (using a graphing calculator or CAS). This restriction means that we must have $-0.15752 < \theta < 1.5507$

Next, we want to determine when the ball hits the ground. This is when
$0 = y(t) = -16t^2 + 176(\sin\theta)t + 10$
We solve this equation using the quadratic formula to get

$$t = \frac{-176\sin\theta \pm \sqrt{176^2\sin^2\theta + 640}}{-32}$$

We are interested in the positive solution, so

$$t = \frac{176\sin\theta + \sqrt{176^2\sin^2\theta + 640}}{32}$$

Substituting this in to $x(t) = 176(\cos\theta)t$ gives

$$x = 44\cos\theta\left(22\sin\theta + \sqrt{484\sin^2\theta + 10}\right)$$

We want to determine the values of θ that ensure that $x < 60$. Using a graphing calculator or a CAS gives $\theta < -0.13429$

Putting together our two conditions on θ now gives the possible range of angles for which the serve will be in:
$-0.15752 < \theta < -0.13429$

20. Repeat example 5.5 with an initial speed of 170 ft/s. By trial and error, find the smallest and largest initial speeds for which the serve will be in.

In these tennis problems, the issue is purely geometric. Time is irrelevant. One can obtain valuable information by eliminating time and writing y as a function of x. For example, with service angle of θ (in degrees below the horizontal), initial speed v_0, and initial height h, one has
$y(t) = -16t^2 - tv_0\sin\theta + h$,
$x(t) = tv_0\cos\theta$, and hence

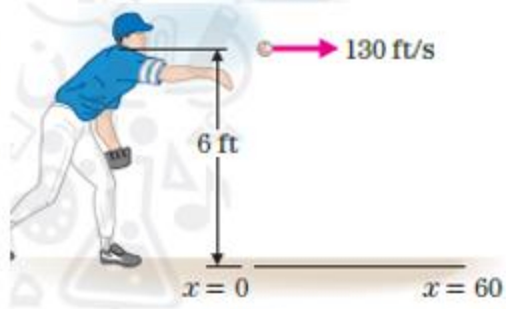
$$y = f(x) = \frac{-16x^2}{v_0^2\cos^2\theta} - \frac{x\sin\theta}{\cos\theta} + h$$

Now one could put $x = 60$ (the serve would be in if $f(60) < 0$), or put $x = 39$ (the serve would clear the net if $f(39) > 3$. If one were to set $f(60) = 0$ and solve for v_0, one would obtain a critical speed (call it v_1) for the given (h, θ), *above which the serve would be out.* Solving $f(39) = 3$ one would obtain a second critical speed (call it v_2), *below which the serve would hit the net.* Below we tabulate v_1 and v_2 for $h = 10$ and selected values of θ.

In the 7° line, we see that it would be necessary to reduce the service speed to 149ft./sec. to get it in, and the net would not be a problem. The 7.6° line has these interesting features: the service at 176 ft./sec. is out, whereas the service at 170 ft./sec. is in.

h	θ	v_1	v_2
feet	degrees	ft/sec	ft/sec
10	7.0	149.0	105.7
10	7.6	171.5	117.4
10	8.0	193.6	127.8

21. A baseball pitcher releases the ball horizontally from a height of 60 ft with an initial speed of 130 ft/s. Find the height of the ball when it reaches home plate 60 ft away. (Hint: Determine the time of flight from the x-equation, then use the y-equation to determine the height.)



Let $(x(t), y(t))$ be the trajectory. In this case
$y(0) = 6, x(0) = 0$
$y'(0) = 0, x'(0) = 130$
$x''(t) \equiv 0, x'(t) \equiv 130$
$x(t) = 130t$
This is 60 at time $t = 6/13$. Meanwhile,
$y''(t) = -32, y'(t) = -32t$
$y(t) = -16t^2 + 6$
$$y\left(\frac{6}{13}\right) = -16\left(\frac{6}{13}\right)^2 + 6 = \frac{438}{169}$$
$$y\left(\frac{6}{13}\right) \approx 2.59 \text{ ft}$$

22. Repeat exercise 21 with an initial speed of 80 ft/s. (Hint: Carefully interpret the negative answer.)

If the initial speed is now 80 ft/s, the equations become
$x(t) = 80t$
$y(t) = -16t^2 + 6$

The ball crosses home plate when $x = 60$, or when $t = 3/4$. At the home plate, we then have,
$y(3/4) = -16(3/4)^2 + 6 = -3$
In other words, the ball is "under" the ground and the ball hits the ground before reaching

23. A baseball player throws a ball toward first base 120 ft away. The ball is released from a height of 5 ft with an initial speed of 120 ft/s at an angle of 5° above the horizontal. Find the height of the ball when it reaches first base.

Let $(x(t), y(t))$ be the trajectory. In this case 5° is converted to $\pi/36$ radians.
$y(0) = 5, x(0) = 0$
$y'(0) = 120 \sin \frac{\pi}{36} \approx 10.46$
$x'(0) = 120 \cos \frac{\pi}{36} \approx 119.54$
$x''(0) \equiv 0$
$x'(t) \equiv 119.54$
$x(t) = 119.54t$
This is 120 when
$t = 120/119.54 = 1.00385\ldots$

Meanwhile,
$y''(t) = -32$
$y'(t) = -32t + 10.46$
$y(t) = -16t^2 + 10.46t + 5$
$y(1.00385) = -16(1.00385)^2$
$+ 10.46(1.00385) + 5$
$y(1.00385) \approx -.62$ ft

$\int x^r dx = \frac{x^{r+1}}{r+1} + c,$ for $r \neq -1$ (power rule)	$\int \frac{1}{x} dx = \ln\|x\| + c,$ for $x \neq 0$
$\int \sin x\, dx = -\cos x + c$	$\int \cos x\, dx = \sin x + c$
$\int \sec^2 x\, dx = \tan x + c$	$\int \sec x \tan x\, dx = \sec x + c$
$\int \csc^2 x\, dx = -\cot x + c$	$\int \csc x \cot x\, dx = -\csc x + c$
$\int e^x\, dx = e^x + c$	$\int e^{-x}\, dx = -e^{-x} + c$
$\int \tan x\, dx = -\ln\|\cos x\| + c$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$	$\int \frac{1}{\|x\|\sqrt{x^2-1}} dx = \sec^{-1} x + c$

$\int \tan x\, dx = -\ln|\cos x| + c = \ln|\sec x| + c \Rightarrow \int \tan(ax+b)\, dx = -\frac{1}{a}\ln|\cos(ax+b)| + c = \frac{1}{a}\ln|\sec(ax+b)| + c$

$\int \cot x\, dx = \ln|\sin x| + c \Rightarrow \int \cot(ax+b)\, dx = \frac{1}{a}\ln|\sin(ax+b)| + c$

$\int \sec x\, dx = \ln|\sec x + \tan x| + c \Rightarrow \int \sec(ax+b)\, dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + c$

$\int \csc x\, dx = \ln|\csc x - \cot x| + c \Rightarrow \int \csc(ax+b)\, dx = \frac{1}{a}\ln|\csc(ax+b) - \cot(ax+b)| + c$

$\int \sec^2 x\, dx = \tan x + c \Rightarrow \int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c$

$\int \csc^2 x\, dx = -\cot x + c \Rightarrow \int \csc^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + c$

$\int \sec x.\tan x\, dx = \sec x + c \Rightarrow \int \sec(ax+b).\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + c$

$$\int \frac{1}{m^2+x^2} dx = \frac{1}{m}.\tan^{-1}\left(\frac{x}{m}\right) + c$$

$$\int \frac{1}{c+bx+ax^2} dx =$$

$$\int \frac{1}{\sqrt{m^2-x^2}} dx = \sin^{-1}\left(\frac{x}{m}\right) + c$$

$$\int \frac{1}{\sqrt{c+bx-ax^2}} dx =$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

$$\int \frac{1}{2\sqrt{u}} du = \sqrt{u} + c$$

Q	Learning Outcome***	Example/Exercise	Page
12	Compute integrals using direct computation and rules	(3-10)	489
13	Compute various interlays using integration by substitution	(14,16,19-24)	489
14	Compute integrals using completing a square before integrating	(11-13)	489

evaluate the integral.

3. $$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\left(\frac{1}{a}\right)dx$$

Let $u=\frac{x}{a}, du=\frac{1}{a}dx$.

$$=\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u)+c$$

$$=\sin^{-1}\left(\frac{x}{a}\right)+c, a>0.$$

4. $$\int \frac{b}{|x|\sqrt{x^2-a^2}}dx$$

$$=\int \frac{b}{|x|\sqrt{\left(\frac{x}{a}\right)^2-1}}\left(\frac{1}{a}\right)dx$$

Let $u=\frac{x}{a}, du=\frac{1}{a}dx$ and $|au|=|x|$.

$$=\int \frac{b}{|au|\sqrt{u^2-1}}du$$

$$=\frac{b}{|a|}\int \frac{1}{|u|\sqrt{u^2-1}}du$$

$$=\frac{b}{|a|}\sec^{-1}(u)+c$$

$$=\frac{b}{|a|}\sec^{-1}\left(\frac{x}{a}\right)+c, a>0.$$

5. $$\int \sin(6t)dt = -\frac{1}{6}\cos(6t)+c$$

6. $$\int \sec 2t \tan 2t \, dt = \frac{1}{2}\sec 2t + c$$

7. $$\int (x^2+4)^2 dx = \int (x^4+8x^2+16)dx$$

$$=\frac{x^5}{5}+\frac{8}{3}x^3+16x+c$$

8. $$\int x(x^2+4)^2 dx = \int (x^5+8x^3+16x)dx$$

$$=\frac{x^6}{6}+2x^4+8x^2+c$$

9. $$\frac{3}{16+x^2}dx = \frac{3}{4}\tan^{-1}\frac{x}{4}+c$$

10. $$\frac{2}{4+4x^2}dx = \frac{1}{2}\tan^{-1}x+c$$

11. $\int \frac{1}{\sqrt{3-2x-x^2}}dx$

$= \int \frac{1}{\sqrt{4-(x+1)^2}}dx = \arcsin\left(\frac{x+1}{2}\right) + c$

12. $\int \frac{x+1}{\sqrt{3-2x-x^2}}dx$

$= -\frac{1}{2}\int \frac{-2(x+1)}{\sqrt{4-(x+1)^2}}dx$

$= -\frac{1}{2}\cdot 2[4-(x+1)^2]^{1/2} + C$

$= -\sqrt{4-(x+1)^2} + c$

13. $\int \frac{4}{5+2x+x^2}dx$

$4\int \frac{1}{4+(x+1)^2}dx = 2\tan^{-1}\left(\frac{x+1}{2}\right) + c$

14. $\int \frac{4x+4}{5+2x+x^2}dx$

$= 2\int \frac{2(x+1)}{4+(x+1)^2}dx = 2\ln|4+(x+1)^2| + c$

16. $\int \frac{t+1}{t^2+2t+4}dt = \int \frac{2(t+1)}{(t+1)^2+3}dt$

$= \frac{1}{2}\ln\left|(t+1)^2+3\right| + c$

19. $\int \frac{4}{x^{1/3}(1+x^{2/3})}\,dx$

Let $u = 1+x^{2/3}$, $du = \frac{2}{3}x^{-1/3}dx$

$\int \frac{4}{x^{1/3}(1+x^{2/3})}dx = 4\left(\frac{3}{2}\right)\int u^{-1}du$

$= 6\ln|u| + C = 6\ln|1+x^{2/3}| + c$

20. $\int \frac{2}{x^{1/4}+x}\,dx$

Let $u = 1+x^{3/4}$, $du = \frac{3}{4}x^{-1/4}dx$

$\int \frac{2}{x^{1/4}+x}dx = \int \frac{2}{x^{1/4}(1+x^{3/4})}dx$

$= 2\left(\frac{4}{3}\right)\int u^{-1}du = \frac{8}{3}\ln|u| + C$

$= \frac{8}{3}\ln|1+x^{3/4}| + c$

21. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u\,du$

$= -2\cos u + C = -2\cos\sqrt{x} + c$

22. $\int \frac{\cos(1/x)}{x^2}\,dx$

Let $u=\frac{1}{x}, du=-\frac{1}{x^2}dx$

$\int \frac{\cos(1/x)}{x^2}dx=-\int \cos u\,du$

$=-\sin u+C=-\sin\frac{1}{x}+c$

23. $\int_0^{\pi} \cos x e^{\sin x}\,dx$

Let $u=\sin x$, $du=\cos x dx$

$\int_0^{\pi} \cos x e^{\sin x}dx=\int_0^0 e^u du=0$

24. $\int_0^{\pi/4} \sec^2 x e^{\tan x}\,dx$

Let $u=\tan x$, $du=\sec^2 x dx$ $\quad \int_0^{\pi/2}\sec^2 x e^{\tan x}dx=\int_0^1 e^u du$ $=e^u\Big|_0^1=e-1$

Q	Learning Outcome***	Example/Exercise	Page
15	**Use integration by parts to compute definite and indefinite integrals**	**(1-6,23,24)**	**496**
16	**Use integration by parts to compute definite and indefinite integrals**	**(19-22)**	**496**

evaluate the integral.

1. $\int x\cos x\,dx$

Let $u=x$, $dv=\cos x dx$

$du=dx$, $v=\sin x$.

$\int x\cos x dx=x\sin x-\int \sin x dx$

$=x\sin x+\cos x+c$

2. $\int x\sin 4x\,dx$

Let $u=x$, $dv=\sin 4x dx$

$du=dx, v=-\frac{1}{4}\cos 4x$

$\int x\sin 4x\,dx$

$=-\frac{1}{4}x\cos 4x-\int -\frac{1}{4}\cos 4x\,dx$

$=-\frac{1}{4}x\cos 4x+\frac{1}{16}\sin 4x+c.$

3. $\int xe^{2x}dx$

Let $u=x$, $dv=e^{2x}dx$

$du=dx,\ v=\frac{1}{2}e^{2x}$.

$\int xe^{2x}dx=\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}dx$

$=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c.$

4. $\int x\ln x\,dx$

Let $u=\ln x$, $dv=x\,dx$

$du=\frac{1}{x}dx$ and $v=\frac{x^2}{2}$.

$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x\,dx$

$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c.$

5. $\int x^2 \ln x \, dx$

Let $u = \ln x$, $dv = x^2 dx$

$du = \frac{1}{x}dx, v = \frac{1}{3}x^3.$

$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c.$

6. $\int \frac{\ln x}{x} dx$

Let $u = \ln x, du = \frac{1}{x} \, dx.$

$\int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + c = \frac{1}{2}(\ln x)^2 + c.$

19. $\int_0^1 x \sin 2x \, dx$

Let $u = x$, $dv = \sin 2x dx$

$du = dx, \; v = -\frac{1}{2}\cos 2x$

$\int_0^1 x \sin 2x dx$

$= -\frac{1}{2}x\cos 2x \Big|_0^1 - \int_0^1 \left(-\frac{1}{2}\cos 2x\right) dx$

$= -\frac{1}{2}(1 \cos 2 - 0 \cos 0) + \frac{1}{2}\int_0^1 \cos 2x dx$

$= -\frac{1}{2}\cos 2 + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^1$

$= -\frac{1}{2}\cos 2 + \frac{1}{4}(\sin 2 - \sin 0)$

$= -\frac{1}{2}\cos 2 + \frac{1}{4}\sin 2$

20. $\int_0^\pi 2x \cos x \, dx$

Let $u = 2x$, $dv = \cos x \, dx$

$du = 2 \, dx$ *and* $v = \sin x.$

$\int_0^\pi 2x \cos x dx = 2x \sin x \,|_0^\pi - 2\int_0^\pi \sin x dx$

$= (2x \sin x + 2\cos x) \,|_0^\pi = -4.$

21. $\int_0^1 x^2 \cos \pi x \, dx$

$$\int_0^1 x^2 \cos \pi x dx$$

Let $u = x^2$, $dv = \cos \pi x dx$,

$$du = 2xdx, v = \frac{\sin \pi x}{\pi}.$$

$$\int_0^1 x^2 \cos \pi x dx = x^2 \frac{\sin \pi x}{\pi}\Big|_0^1 - \int_0^1 \frac{\sin \pi x}{\pi} 2xdx$$

$$= (0-0) - \frac{2}{\pi}\int_0^1 x \sin(\pi x)\, dx$$

$$= -\frac{2}{\pi}\int_0^1 x \sin(\pi x)\, dx$$

Let $u = x$, $dv = \sin(\pi x)dx$,

$$du = dx,\ v = -\frac{\cos(\pi x)}{\pi}.$$

$$-\frac{2}{\pi}\int_0^1 x \sin(\pi x) dx$$

$$= -\frac{2}{\pi}\left\{-\frac{x\cos(\pi x)}{\pi}\Big|_0^1 - \int_0^1 -\frac{\cos(\pi x)}{\pi} dx\right\}$$

$$= -\frac{2}{\pi}\left\{(-\frac{\cos \pi}{\pi} - 0) + \frac{1}{\pi}\left[\frac{\sin(\pi x)}{\pi}\right]_0^1\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{1}{\pi} + \frac{1}{\pi}(0-0)\right\} = -\frac{2}{\pi^2}$$

22. $\int_0^1 x^2 e^{3x} dx$

$$\int_0^1 x^2 e^{3x} dx$$

Let $u = x^2$, $dv = e^{3x}\, dx$,

$$du = 2xdx,\ v = \frac{e^{3x}}{3}.$$

$$\int_0^1 x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3}\Big|_0^1 - \int_0^1 \frac{e^{3x}}{3} 2xdx$$

$$= \frac{1}{3}(e^3 - 0) - \frac{2}{3}\int_0^1 xe^{3x} dx.$$

Let $u = x$, $dv = e^{3x}dx$,

$$dv = dx, v\frac{e^{3x}}{3}.$$

$$\frac{e^3}{3} - \frac{2}{3}\int_0^1 xe^{3x} dx$$

$$= \frac{e^3}{3} - \frac{2}{3}\left\{x\frac{e^{3x}}{3}\Big|_0^1 - \int_0^1 \frac{e^{3x}}{3} dx\right\}$$

$$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \int_0^1 \frac{e^{3x}}{3} dx\right\}$$

$$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \left[\frac{e^{3x}}{9}\right]_0^1\right\}$$

$$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \frac{1}{9}(e^3 - 1)\right\}$$

$$= \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2}{27}(e^3 - 1)$$

$$= \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2e^3}{27} - \frac{2}{27} = \frac{5e^3}{27} - \frac{2}{27}$$

Q	Learning Outcome***	Example/Exercise	Page
17	**Integrate functions of the form sin^n(x) cos^m(x)**	**(1-8)**	**507**
18	**Integrate functions of the form sec^m(x) tan^n(x)**	**(9-16)**	**507**
19	**Integrate trigonometric functions using the substitution x=a sin(y)**	**(21-26)**	**507**
20	**Integrate trigonometric functions using the substitution x=a tan(y)**	**(33-41)**	**507**
21	**Integrate trigonometric functions using the substitution x=a sec(y)**	**(27-32)**	**507**

evaluate the integral.

1. $\int \cos x \sin^4 x \, dx$

Let $u = \sin x$, $du = \cos x dx$

$$\int \cos x \sin^4 x dx = \int u^4 du$$

$$= \frac{1}{5}u^5 + c = \frac{1}{5}\sin^5 x + c$$

2. $\int \cos^3 x \sin^4 x \, dx$

Let $u = \sin x$, $du = \cos x dx$

$$\int \cos^3 x \sin^4 x dx = \int (1-u^2)u^4 du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + c$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

3. $\int_0^{\pi/4} \cos 2x \sin^3 2x \, dx$

Let $u = \sin 2x$, $du = 2\cos 2x dx$.

$$\int_0^{\pi/4} \cos 2x \sin^3 2x dx$$

$$= \frac{1}{2}\int_0^1 u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_0^1 = \frac{1}{8}$$

4. $\int_{\pi/4}^{\pi/3} \cos^3 3x \sin^3 3x \, dx$

Let $u = \cos 3x$, $du = -3\sin x dx$.

$$\int_{\pi/4}^{\pi/3} (\cos^3 3x)(\sin^3 3x) \, dx$$

$$= -\frac{1}{3}\int_{-1/\sqrt{2}}^{-1} u^3(1-u^2) \, du$$

$$= -\frac{1}{3}\left[\frac{u^4}{4} - \frac{u^6}{6}\right]_{\frac{-1}{\sqrt{2}}}^{-1}$$

$$= -\frac{1}{3}\left(\frac{3}{16} - \frac{7}{48}\right) = -\frac{1}{72}$$

5. $\int_0^{\pi/2} \cos^2 x \sin x \, dx$

Let $u = \cos x$, $du = -\sin x dx$

$$\int_0^{\pi 2} \cos^2 x \sin x dx = \int_1^0 u^2(-du)$$

$$= \left(-\frac{1}{3}u^3\right)\Big|_1^0 = \frac{1}{3}$$

6. $\int_{-\pi/2}^{0} \cos^3 x \sin x \, dx$

Let $u = \cos x$, $du = -\sin x dx$

$$\int_{-\pi/2}^{0} \cos^3 x \sin x dx = -\int_0^1 u^3 du = -1$$

7. $\int \cos^2 (x+1) \, dx$

$$\int \cos^2(x+1)dx$$

$$= \frac{1}{2}\int (1+\cos 2(x+1))dx$$

$$= \frac{1}{2}x + \frac{1}{4}(\sin 2(x+1)) + c.$$

8. $\int \sin^4 (x-3) \, dx$

Let $u = x - 3$, $du = dx$

$$\int \sin^4(x-3)dx = \int \sin^4 u du$$

$$= \int (\sin^2 u)^2 du$$

$$= \int \frac{(1-\cos 2u)}{2} \times \frac{(1-\cos 2u)}{2} du$$

$$= \int \frac{1}{4}(1 - 2\cos 2u + \cos^2 2u)$$

$$= \frac{1}{4}\int \left[1 - 2\cos 2u + \frac{1}{2}(1+\cos 4u)\right] du$$

$$= \frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\cos 4u + c$$

$$= \frac{3}{8}(x-3) - \frac{1}{4}\sin 2(x-3)$$

$$+ \frac{1}{32}\cos 4(x-3) + c.$$

9. $\int \tan x \sec^3 x \, dx$

Let $u = \sec x$, $du = \sec x \tan x dx$

$$\int \tan x \sec^3 x dx$$

$$= \int \tan x \sec x \sec^2 x dx$$

$$= \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\sec^3 x + c$$

10. $\int \cot x \csc^4 x \, dx$

Let $u = \cot x$, $du = -\csc^2 x dx$

$$\int \cot x \csc^4 x dx$$

$$= -\int \cot x(1+\cot^2 x)\cdot \csc^2 x dx$$

$$= -\int (u+u^3)du = -\frac{u^2}{2} - \frac{u^4}{4} + C$$

$$= -\frac{\cot^2 x}{2} - \frac{\cot^4}{4} + c$$

11. $\int x\tan^3(x^2+1)\sec(x^2+1)\,dx$

Let $u = x^2 + 1$, so that $du = 2x dx$.

$$\int x\tan^3(x^2+1)(\sec(x^2+1))\,dx$$

$$=\frac{1}{2}\int \tan^3 u(\sec u)\,du$$

$$=\frac{1}{2}\int[(\sec^2 u-1)\tan u(\sec u)]\,du$$

Let $\sec u = t$, $dt = \tan u \sec u du$

$$=\frac{1}{2}\int(t^2-1)\,dt=\frac{1}{2}\left[\frac{t^3}{3}-t\right]+c$$

$$=\frac{1}{2}\left[\frac{\sec^3 u}{3}-\sec u\right]+c$$

$$=\frac{1}{6}\sec^3(x^2+1)-\frac{1}{2}\sec(x^2+1)+c.$$

12. $\int \tan(2x+1)\sec^3(2x+1)\,dx$

Let $u = 2x + 1$, so that $du = 2dx$.

$$\int \tan(2x+1).\sec^3(2x+1)\,dx$$

$$=\frac{1}{2}\int \tan u.\sec u.\sec^2 u\,du$$

$$=\frac{1}{2}\int \sec^2 u\tan u\sec u\,du$$

Let $t = \sec u$, so that $dt = \tan u \sec u du$.

$$=\frac{1}{2}\int t^2 dt=\frac{1}{2}\left[\frac{t^3}{3}\right]+c$$

$$=\frac{1}{2}\left[\frac{\sec^3 u}{3}\right]+c=\frac{1}{6}\sec^3(2x+1)+c.$$

13. $\int \cot^2 x\csc^4 x\,dx$

Let $u = \cot x$, $du = (-\csc^2 x)\,dx$

$$\int \cot^2 x\csc^4 x\,dx=\int \cot^2 x(1+\cot^2 x)\csc^2 x\,dx$$

$$=-\int u^2(1+u^2)du$$

$$-\frac{u^3}{3}-\frac{u^5}{5}+c$$

$$=-\frac{(\cot x)^3}{3}-\frac{(\cot x)^5}{5}+c.$$

14. $\int \cot^2 x\csc^2 x\,dx$

Let $u = \cot x$, $du = (-\csc^2 x)\,dx$.

$$\int \cot^2 x\csc^2 x\,dx=-\int u^2 du$$

$$=-\frac{u^3}{3}+c=\frac{\cot^3 x}{3}+c.$$

Expression	Trigonometric Substitution	Interval	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\theta\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$	$\sec^2\theta-1=\tan^2\theta$

15. $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx$

Let $u = \tan x$, $du = \sec^2 x dx$

$$\int_0^{\pi/4} \tan^4 x \sec^4 x dx$$

$$= \int_0^{\pi/4} \tan^4 x \sec^2 x \sec^2 x dx$$

$$= \int_0^{\pi/4} \tan^4 x(1+\tan^2 x)\sec^2 x dx$$

$$= \int_0^1 u^4(1+u^2)du$$

$$= \int_0^1 (u^4+u^6)du = \left.\frac{u^5}{5}+\frac{u^7}{7}\right|_0^1 = \frac{12}{35}$$

16. $\int_{-\pi/4}^{\pi/4} \tan^4 x \sec^2 x \, dx$

Let $u = \tan x$, $du = \sec^2 x dx$.

$$\int_{\pi/4}^{\pi/4} \tan^4 x \sec^2 x dx$$

$$= \int_{-1}^1 u^4 du = \left.\frac{u^5}{5}\right|_{-1}^1 = \frac{2}{5}$$

21. $\int \frac{1}{x^2\sqrt{9-x^2}} dx$

Let $x = 3\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3 \cos\theta \, d\theta$

$$\int \frac{1}{x^2\sqrt{9-x^2}} dx = \int \frac{3\cos\theta}{9\sin^2\theta \cdot 3\cos\theta} d\theta$$

$$= \frac{1}{9}\int \csc^2\theta d\theta = -\frac{1}{9}\cot\theta + C$$

By drawing a diagram, we see that if $x = \sin\theta$, then $\cot\theta = \frac{\sqrt{9-x^2}}{x}$.

Thus the integral $= -\frac{\sqrt{9-x^2}}{9x} + c$

22. $\int \frac{1}{x^2\sqrt{16-x^2}} dx$

Let $x = 4\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$dx = 4 \cos\theta \, d\theta$

$$\int \frac{1}{x^2\sqrt{16-x^2}} dx = \int \frac{\cos\theta}{16\sin^2\theta\cos\theta} d\theta$$

$$= \frac{1}{16}\int \csc^2\theta d\theta = -\frac{1}{16}\cot\theta + c$$

$$= -\frac{\sqrt{16-x^2}}{16x} + c$$

23. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Let $x = 4\sin\theta$, so that $dx = 4\cos\theta\, d\theta$.

$$\int \frac{x^2}{\sqrt{16-x^2}}dx = \int \frac{(16\sin^2\theta)4\cos\theta}{\sqrt{16-(4\sin\theta)^2}}d\theta$$

$$= 64\int \frac{(\sin^2\theta)\cos\theta}{\sqrt{16-16\sin^2\theta}}d\theta$$

$$= 64\int \frac{(\sin^2\theta)\cos\theta}{4\sqrt{(1-\sin^2\theta)}}d\theta$$

$$= 16\int \frac{\sin^2\theta\cos\theta}{\cos\theta}d\theta = 16\int \sin^2\theta d\theta$$

$$= 16\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 8\left[\int d\theta - \int(\cos 2\theta)d\theta\right]$$

$$= 8\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - 4\sin\left[2\sin^{-1}\left(\frac{x}{4}\right)\right] + c.$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + c$$

24. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

Let $x = 3\sin\theta$, so that $dx = 3\cos\theta\, d\theta$.

$$\int \frac{x^3}{\sqrt{9-x^2}}dx$$

$$= \int \frac{27(\sin^3\theta)}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,d\theta$$

$$= 81\int \frac{\sin^3\theta}{\sqrt{9-9\sin^2\theta}}(\cos\theta)\,d\theta$$

$$= 81\int\left(\frac{\sin^3\theta}{3\cos\theta}\right)\cos\theta d\theta = 27\int \sin^3\theta d\theta$$

$$= 27\int\left(\frac{3\sin\theta - \sin 3\theta}{4}\right)d\theta$$

$$= \frac{27}{4}\left[3\int \sin\theta d\theta - \int \sin 3\theta d\theta\right]$$

$$= \frac{27}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right] + c$$

$$= \frac{27}{4}\left\{-3\cos\left[\sin^{-1}\left(\frac{x}{3}\right)\right] + \frac{\cos[\sin^{-1}(\frac{x}{3})]}{3}\right\} + c.$$

25. $\int_0^2 \sqrt{4-x^2}\,dx$

This is the area of a quarter of a circle of radius 2,

$$\int_0^2 \sqrt{4-x^2}dx = \pi$$

26. $\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2$, $du = -2x dx$

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}dx = -\int_4^3 \frac{du}{2\sqrt{u}}$$

$$= -u^{1/2}\Big|_4^3 = 2 - \sqrt{3}$$

27. $\int \frac{x^2}{\sqrt{x^2-9}}\,dx$

Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$.

$$I = \int \frac{x^2}{\sqrt{x^2-9}}dx$$
$$= \int \frac{27\sec^2\theta\sec\theta\tan\theta}{\sqrt{9\sec^2\theta-9}}d\theta$$
$$= \int 9\sec^3\theta\,d\theta$$

Use integration by parts.
Let $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$. This gives

$$\int \sec^3\theta\,d\theta$$
$$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$
$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta-1)\,d\theta$$
$$= \sec\theta\tan\theta + \int \sec\theta\,d\theta - \int \sec^3\theta\,d\theta$$
$$2\int \sec^3\theta\,d\theta$$
$$= \sec\theta\tan\theta + \int \sec\theta\,d\theta$$
$$\int \sec^3\theta\,d\theta$$
$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta$$

This leaves us to compute $\int \sec\theta\,d\theta$.
For this notice if $u = \sec\theta + \tan\theta$ then $du = \sec\theta\tan\theta + \sec^2\theta$.

$$\int \sec\theta\,d\theta$$
$$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\,d\theta$$
$$= \int \frac{1}{u}du = \ln|u| + c$$
$$= \ln|\sec\theta+\tan\theta| + c$$

Putting all these together and using $\sec\theta = \frac{x}{3}$, $\tan\theta = \frac{\sqrt{x^2-9}}{3}$:

$$\int \frac{x^2}{\sqrt{x^2-9}}dx = \int 9\sec^3\theta\,d\theta$$
$$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\int \sec\theta\,d\theta$$
$$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta| + c$$
$$= \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{x^2-9}}{3}\right) + \frac{9}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + c$$
$$= \frac{x\sqrt{x^2-9}}{2} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + c$$

28. $\int x^3\sqrt{x^2-1}\,dx$

Let $u = x^2 - 1$, $du = 2x\,dx$

$$\int x^3\sqrt{x^2-1}dx$$
$$= \frac{1}{2}\int x^2\sqrt{x^2-1}(2x)dx$$
$$= \frac{1}{2}\int (u+1)\sqrt{u}du$$
$$= \frac{1}{2}\int u^{3/2} + u^{1/2}\,du$$
$$= \frac{1}{2}\left(\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right) + c$$
$$= \frac{1}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + c$$

29. $\int \frac{2}{\sqrt{x^2-4}}\,dx$

Let $x = 2\sec\theta, dx = 2\sec\theta\tan\theta d\theta$

$$\int \frac{2}{\sqrt{x^2-4}}dx = \int \frac{4\sec\theta\tan\theta}{2\tan\theta}d\theta$$

$$= 2\int \sec\theta d\theta$$

$$= 2\ln|2\sec\theta + 2\tan\theta| + c$$

$$= 2\ln\left|x + \sqrt{x^2-4}\right| + c$$

30. $\int \frac{x}{\sqrt{x^2-4}}\,dx$

Let $x = 2\sec\theta, dx = 2\sec\theta\tan\theta d\theta$

$$\int \frac{x}{\sqrt{x^2-4}}dx = \int \frac{4\sec^2\theta\tan\theta}{2\tan\theta}d\theta$$

$$= 2\int \sec^2\theta d\theta = 2\tan\theta + C = \sqrt{x^2-4} + c$$

31. $\int \frac{\sqrt{4x^2-9}}{x}\,dx$

$$\int \frac{\sqrt{4x^2-9}}{x}dx = \int \frac{\sqrt{4x^2-9}}{4x^2}4xdx$$

Let $u = \sqrt{4x^2-9}$,

$$du = \frac{1}{2\sqrt{4x^2-9}}8xdx = \frac{1}{2u}8xdx$$

or $udu = 4xdx$.

Hence, we have

$$\int \frac{\sqrt{4x^2-9}}{x}dx$$

$$= \int \frac{u}{u^2+9}udu = \int \frac{u^2}{u^2+9}du$$

$$= \int \frac{u^2+9-9}{u^2+9}du = \int du - \int \frac{9}{u^2+9}du$$

$$= u - 9\tan^{-1}\left(\frac{u}{3}\right) + c$$

$$= \sqrt{4x^2-9} - 9\tan^{-1}\left(\frac{\sqrt{4x^2-9}}{3}\right) + c.$$

32. $\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Let $x = 2\sec\theta, dx = 2\tan\theta\sec\theta d\theta$.

$$\int \frac{\sqrt{x^2-4}}{x^2}dx$$

$$= \int \frac{\sqrt{4\sec^2\theta-4}}{4\sec^2\theta}(2\tan\theta\sec\theta)\,d\theta$$

$$= \int \frac{2\tan\theta}{4\sec^2\theta}(2\tan\theta\sec\theta)\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta-1}{\sec\theta}d\theta$$

$$= \int \sec\theta d\theta - \int \frac{1}{\sec\theta}d\theta$$

$$= \int \sec\theta d\theta - \int \cos\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c$$

$$= \ln\left|\sec\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + \tan\left[\sec^{-1}\left(\frac{x}{2}\right)\right]\right|$$
$$- \sin\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + c$$

$$= \ln\left|\left(\frac{x}{2}\right) + \tan\left[\sec^{-1}\left(\frac{x}{2}\right)\right]\right|$$
$$- \sin\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + c.$$

33. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$

Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{9+x^2}}\,dx$$
$$= \int \frac{27\tan^2\theta\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta$$
$$= \int 9\tan^2\theta\sec\theta\, d\theta$$
$$= 9\int (\sec^2\theta - 1)\sec\theta\, d\theta$$
$$= 9\int \sec^3\theta\, d\theta - 9\int \sec\theta\, d\theta$$
$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + c$$
$$= \frac{9}{2}\left(\frac{\sqrt{9+x^2}}{3}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c$$
$$= \frac{x\sqrt{9+x^2}}{2} - \frac{9}{2}\ln\left|\frac{x+\sqrt{9+x^2}}{3}\right| + c$$

34. $\int x^3\sqrt{8+x^2}\,dx$

Let $x = 2\sqrt{2}\tan\theta, dx = 2\sqrt{2}\sec^2\theta\, d\theta$

$$\int x^3\sqrt{8+x^2}\,dx$$
$$= \int (16\sqrt{2}\tan^3\theta)(2\sqrt{2}\sec\theta)\,d\theta$$
$$= 64\int \tan^3\theta\sec\theta\, d\theta$$
$$= 64\int (\sec^2\theta - 1)(\sec\theta\tan\theta\, d\theta)$$
$$= 64\int (u^2 - 1)\,du = \frac{64}{3}u^3 - 64u + c$$
$$= \frac{64}{3}\sec^3\theta - 64\sec\theta + c$$
$$= \frac{64}{3}\left(\frac{\sqrt{8+x^2}}{2\sqrt{2}}\right)^3 - 64\left(\frac{\sqrt{8+x^2}}{2\sqrt{2}}\right) + c$$
$$= \frac{2\sqrt{2}}{3}(8+x^2)^{3/2} - 16\sqrt{2}(8+x^2)^{1/2} + c$$

35. $\int \sqrt{16+x^2}\,dx$

Let $x = 4\tan\theta$, $dx = 4\sec^2\theta\, d\theta$

$$\int \sqrt{16+x^2}\,dx$$
$$= \int \sqrt{16+16\tan^2\theta}\cdot 4\sec^2\theta\, d\theta$$
$$= 16\int \sec^3\theta\, d\theta$$
$$= 16\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right)$$
$$= 8\sec\theta\tan\theta + 8\int \sec\theta\, d\theta$$
$$= 8\sec\theta\tan\theta + 8\ln|\sec\theta + \tan\theta| + c$$
$$= \frac{1}{2}x\sqrt{16+x^2}$$
$$+8\ln\left|\frac{1}{4}\sqrt{16+x^2} + \frac{x}{4}\right| + c$$

36. $\int \frac{1}{\sqrt{4+x^2}}\,dx$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$
$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$$
$$= \ln\left|\frac{x+\sqrt{4+x^2}}{2}\right| + c$$

37. $\int_0^1 x\sqrt{x^2+8}\,dx$

Let $u = x^2 + 8$, $du = 2x\,dx$

$$\int_0^1 x\sqrt{x^2+8}\,dx = \frac{1}{2}\int_8^9 u^{1/2}\,du$$
$$= \frac{1}{3}u^{3/2}\Big|_8^9 = \frac{27-16\sqrt{2}}{3}$$

41. $\int \frac{x}{\sqrt{x^2+4x}}\,dx$

$$\int \frac{x}{\sqrt{x^2+4x}}\,dx$$
$$= \frac{1}{2}\int \frac{2x+4-4}{\sqrt{x^2+4x}}\,dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+4x}}\,dx - \frac{1}{2}\int \frac{4}{\sqrt{x^2+4x}}\,dx$$

Let $u = x^2 + 4x$, $du = (2x+4)\,dx$.

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} - \frac{1}{2}\int \frac{4}{\sqrt{x^2+4x-4+4}}\,dx$$
$$= u^{1/2} - \frac{1}{2}\int \frac{4}{\sqrt{(x+2)^2-4}}\,dx$$
$$= \sqrt{(x^2+4x)}$$
$$-2\log\left[(x^2+4x) + \sqrt{(x+2)^2-4}\right] + c$$

38. $\int_0^2 x^2\sqrt{x^2+9}\,dx$

Let $= 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$$I=\int x^2\sqrt{x^2+9}dx$$
$$=\int 27\tan^2\theta\sec^2\theta\sqrt{9\tan^2\theta+9}dx$$
$$=81\int\tan^2\theta\sec^3\theta dx$$
$$=81\int(\sec^2\theta-1)\sec^3\theta dx$$
$$=81\int(\sec^5\theta-\sec^3\theta)dx$$

To compute $\int\sec^5\theta\, d\theta$, we use integration by arts with $u=\sec^3\theta$ and $dv=\sec^2\theta d\theta$.

$$\int\sec^5\theta\, d\theta$$
$$=\sec^3\theta\tan\theta-\int 3\sec^3\theta\tan^2\theta d\theta$$
$$=\sec^3\theta\tan\theta-3\int\sec^3\theta(\sec^2\theta-1)d\theta$$
$$=\sec^3\theta\tan\theta-3\int(\sec^5\theta-\sec^3\theta)d\theta$$
$$4\int\sec^5\theta d\theta$$
$$=\sec^3\theta\tan\theta+3\int\sec^3\theta d\theta\int\sec^5\theta d\theta$$
$$=\frac{1}{4}\sec^3\theta\tan\theta+\frac{3}{4}\int\sec^3\theta d\theta$$

To compute $\int\sec^3\theta d\theta$ and $\int\sec\theta d\theta$, see Exercise 27.

Putting all this together gives:

$$I=81\int(\sec^5\theta-\sec^3\theta)dx$$
$$=\frac{81}{4}\sec^3\theta\tan\theta+\frac{243}{4}\int\sec^3\theta d\theta$$
$$-81\int\sec^3\theta d\theta$$
$$=\frac{81}{4}\sec^3\theta\tan\theta-\frac{81}{4}\int\sec^3\theta d\theta$$
$$=\frac{81}{4}\sec^3\theta\tan\theta-\frac{81}{8}\sec\theta\tan\theta$$
$$-\frac{81}{8}\ln|\sec\theta+\tan\theta|+c$$

We don't worry about the result being in terms of x since this is a definite integral. Our limits of integration are $x=0$ and $x=2$. In terms of θ this means the limits of integration correspond to $\theta=0$ and $\tan\theta=\frac{2}{3}$.

$$\int_0^2 x^2\sqrt{x^2+9}dx$$
$$=\left(\frac{81}{4}\sec^3\theta\tan\theta-\frac{81}{8}\sec\theta\tan\theta\right.$$
$$\left.-\frac{81}{8}\ln|\sec\theta+\tan\theta|\right)\Bigg|_{x=0}^{x=2}$$
$$=\left(\frac{81}{4}\left[\frac{\sqrt{13}}{3}\right]^3\left(\frac{2}{3}\right)-\frac{81}{8}\left(\frac{\sqrt{13}}{3}\right)\left(\frac{2}{3}\right)\right.$$
$$\left.-\frac{81}{8}\ln\left|\frac{\sqrt{13}}{3}+\frac{2}{3}\right|\right)$$
$$-\left(\frac{81}{4}(1)(0)-\frac{81}{8}(1)(0)-\frac{81}{8}\ln|1+0|\right)$$
$$=\frac{17\sqrt{13}}{4}-\frac{81}{8}\ln\left|\frac{2+\sqrt{13}}{3}\right|$$

39. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$.

$$\int \frac{x^3}{\sqrt{1+x^2}}dx = \int\left(\frac{\tan^3\theta}{\sec\theta}\right)\sec^2\theta d\theta$$

$$= \int (\tan^2\theta)(\tan\theta\sec\theta)d\theta$$

Let $t = \sec\theta$, $dt = \tan\theta\sec\theta\, d\theta$.

$$= \int (\sec^2\theta - 1)\tan\theta\sec\theta d\theta$$

$$= \int (t^2-1)dt = \left[\frac{t^3}{3} - t\right] + c$$

$$= \left[\frac{\sec^3\theta}{3} - \sec\theta\right] + c$$

$$= \left[\frac{\sec^3(\tan^{-1}x)}{3} - \sec(\tan^{-1}x)\right] + c.$$

40. $\int \frac{x+1}{\sqrt{4+x^2}}\,dx$

Let $x = 2\tan\theta$, $d\theta = (2\sec^2\theta)\,d\theta$.

$$\int \frac{x+1}{\sqrt{4+x^2}}dx$$

$$= \int\left(\frac{2\tan\theta+1}{\sqrt{4+4\tan^2\theta}}\right)2\sec^2\theta d\theta$$

$$= \int\left(\frac{2\tan\theta+1}{2\sec\theta}\right)(2\sec^2\theta)d\theta$$

$$= \int (2\tan\theta+1)(\sec\theta)d\theta$$

$$= 2\int \sec\theta\tan\theta d\theta + \int \sec\theta d\theta$$

$$= 2\sec\theta + \ln|\sec\theta + \tan\theta| + c$$

$$= 2\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \ln\left|\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \tan\left[\tan^{-1}\left(\frac{x}{2}\right)\right]\right| + c$$

$$= 2\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \ln\left|\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \left(\frac{x}{2}\right)\right| + c.$$

Q	Learning Outcome***	Example/Exercise	Page
22	**Integrate rational functions using partial fractions in different cases**	**(1-12)**	**516**

In exercises 1–20, find the partial fractions decomposition and an antiderivative. If you have a CAS available, use it to check your answer.

1. $\frac{x-5}{x^2-1}$

$$\frac{x-5}{x^2-1}=\frac{x-5}{(x+1)(x-1)}$$
$$=\frac{A}{x+1}+\frac{B}{x-1}$$
$$x-5=A(x-1)+B(x+1)$$
$$x=-1:-6=-2A;A=3$$
$$x=1:-4=2B;B=-2$$
$$\frac{x-5}{x^2-1}=\frac{3}{x+1}-\frac{2}{x-1}$$
$$\int\frac{x-5}{x^2-1}dx=\int\left(\frac{3}{x+1}-\frac{2}{x-1}\right)dx$$
$$=3\ln|x+1|-2\ln|x-1|+c$$

2. $\frac{5x-2}{x^2-4}$

$$\frac{5x-2}{x^2-4}=\frac{5x-2}{(x+2)(x-2)}$$
$$=\frac{A}{x+2}+\frac{B}{x-2}$$
$$5x-2=A(x-2)+B(x+2)$$
$$x=-2:-12=-4A;A=3$$
$$x=2:8=4B;B=2$$
$$\frac{5x-2}{x^2-4}=\frac{3}{x+2}+\frac{2}{x-2}$$
$$\int\frac{5x-2}{x^2-4}dx=\int\left(\frac{3}{x+2}+\frac{2}{x-2}\right)dx$$
$$=3\ln|x+2|+2\ln|x-2|+c$$

3. $\frac{6x}{x^2-x-2}$

$$\frac{6x}{x^2-x-2}=\frac{6x}{(x-2)(x+1)}$$
$$=\frac{A}{x-2}+\frac{B}{x+1}$$
$$6x=A(x+1)+B(x-2)$$
$$x=2:12=3A;A=4$$
$$x=-1:-6=-3B;B=2$$
$$\frac{6x}{x^2-x-2}=\frac{4}{x-2}+\frac{2}{x+1}$$
$$\int\frac{6x}{x^2-x-2}dx$$
$$=\int\left(\frac{4}{x-2}+\frac{2}{x+1}\right)dx$$
$$=4\ln|x-2|+2\ln|x+1|+c$$

4. $\frac{3x}{x^2-3x-4}$

$$\frac{3x}{x^2-3x-4}=\frac{3x}{(x+1)(x-4)}$$
$$=\frac{A}{x+1}+\frac{B}{x-4}$$
$$3x=A(x-4)+B(x+1)$$
$$x=-1:-3=-5A;A=\frac{3}{5}$$
$$x=3:12=5B;B=\frac{12}{5}$$
$$\frac{3x}{x^2-3x-4}=\frac{3/5}{x+1}+\frac{12/5}{x-4}$$
$$\int\frac{3x}{x^2-3x-4}dx$$
$$=\int\left(\frac{3/5}{x+1}+\frac{12/5}{x-4}\right)dx$$
$$=\frac{3}{5}\ln|x+1|+\frac{12}{5}\ln|x-4|+c$$

5. $\frac{-x+5}{x^3-x^2-2x}$

$$\frac{-x+5}{x^3-x^2-2x}=\frac{-x+5}{x(x-2)(x+1)}$$
$$=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$$
$$-x+5=A(x-2)(x+1)+Bx(x+1)+cx(x-2)$$
$$x=0:5=-2A:A=-\frac{5}{2}$$
$$x=2:3=6B:B=\frac{1}{2}$$
$$x=-1:6=3C:C=2$$
$$\frac{-x+5}{x^3-x^2-2x}=-\frac{5/2}{x}+\frac{1/2}{x-2}+\frac{2}{x+1}$$
$$\int\frac{-x+5}{x^3-x^2-2x}dx$$
$$=\int\left(-\frac{5/2}{x}+\frac{1/2}{x-2}+\frac{2}{x+1}\right)dx$$
$$=-\frac{5}{2}\ln|x|+\frac{1}{2}\ln|x-2|+2\ln|x+1|+c$$

6. $\frac{3x+8}{x^3+5x^2+6x}$

$$\frac{3x+8}{x^3+5x^2+6x}=\frac{3x+8}{x(x+2)(x+3))}$$
$$=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+3}$$
$$3x+8=A(x+2)(x+3)+Bx(x+3)+cx(x+2)$$
$$x=0:8=6A;A=\frac{4}{3}$$
$$x=-2:2=-2B;B=-1$$
$$x=-3:-1=3C;C=-\frac{1}{3}$$
$$\frac{3x+8}{x^3+5x^2+6x}=\frac{4/3}{x}-\frac{1}{x+2}-\frac{1/3}{x+3}$$
$$\int\frac{3x+8}{x^3+5x^2+6x}dx$$
$$=\int\left(\frac{4/3}{x}-\frac{1}{x+2}-\frac{1/3}{x+3}\right)dx$$
$$=\frac{4}{3}\ln|x|-\ln|x+2|-\frac{1}{3}\ln|x+3|+c$$

7. $\frac{5x-23}{6x^2-11x-7}$

$$\frac{5x-23}{6x^2-11x-7}=\frac{5x-23}{(2x+1)(3x-7)}$$
$$=\frac{A}{2x+1}+\frac{B}{3x-7}$$
$$5x-23=A(3x-7)+B(2x+1)$$
$$x=-\frac{1}{2}:-\frac{51}{2}=-\frac{17}{2}A;A=3$$
$$x=\frac{7}{3}:-\frac{34}{3}=\frac{17}{3}B;B=-2$$
$$\frac{5x-23}{6x^2-11x-7}=\frac{3}{2x+1}-\frac{2}{3x-7}$$
$$\int\frac{5x-23}{6x^2-11x-7}dx$$
$$=\int\left(\frac{3}{2x+1}-\frac{2}{3x-7}\right)dx$$
$$=\frac{3}{2}\ln|2x+1|-\frac{2}{3}\ln|3x-7|+c$$

8. $\frac{3x+5}{5x^2-4x-1}$

$$\frac{3x+5}{5x^2-4x-1}=\frac{3x+5}{(5x+1)(x-1)}$$
$$=\frac{A}{5x+1}+\frac{B}{x-1}$$
$$3x+5=A(x-1)+B(5x+1)$$
$$x=-\frac{1}{5}:\frac{22}{5}=-\frac{6}{5}A;A=-\frac{11}{3}$$
$$x=1:8=6B;B=\frac{4}{3}$$
$$\frac{3x+5}{5x^2-4x-1}=-\frac{11/3}{5x+1}+\frac{4/3}{x-1}$$
$$\int\frac{3x+5}{5x^2-4x-1}dx$$
$$=\int\left(-\frac{11/3}{5x+1}+\frac{4/3}{x-1}\right)dx$$
$$=-\frac{11}{15}\ln|5x+1|+\frac{4}{3}\ln|x-1|+c$$

9. $\dfrac{x-1}{x^3+4x^2+4x}$

$$\frac{x-1}{x^3+4x^2+4x}=\frac{x-1}{x(x+2)^2}$$
$$=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$
$$x-1=A(x+2)^2+Bx(x+2)+Cx$$
$$x=0:-1=4A;A=-\frac{1}{4}$$
$$x=-2:-3=-2C;C=\frac{3}{2}$$
$$x=1:0=9A+3B+C;B=\frac{1}{4}$$
$$\frac{x-1}{x^3+4x^2+4x}$$
$$=-\frac{1/4}{x}+\frac{1/4}{x+2}+\frac{3/2}{(x+2)^2}$$
$$\int\frac{x-1}{x^3+4x^2+4x}dx$$
$$=\int\left(-\frac{1/4}{x}+\frac{1/4}{x+2}+\frac{3/2}{(x+2)^2}\right)dx$$
$$=-\frac{1}{4}\ln|x|+\frac{1}{4}\ln|x+2|-\frac{3}{2(x+2)}+c$$

10. $\dfrac{4x-5}{x^3-3x^2}$

$$\frac{4x-5}{x^3-3x^2}=\frac{4x-5}{x^2(x-3)}$$
$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}$$
$$4x-5=Ax(x-3)+B(x-3)+Cx^2$$
$$=(A+C)x^2+(-3A+B)x+(-3B)$$
$$B=\frac{5}{3};A=-\frac{7}{9};C=\frac{7}{9}$$
$$\frac{4x-5}{x^3-3x^2}=-\frac{7/9}{x}+\frac{5/3}{x^2}+\frac{7/9}{x-3}$$
$$\int\frac{4x-5}{x^3-3x^2}dx$$
$$=\int\left(-\frac{7/9}{x}+\frac{5/3}{x^2}+\frac{7/9}{x-3}\right)dx$$
$$=-\frac{7/9}{\ln}|x|-\frac{5}{3}\frac{1}{x}+\frac{7}{9}\ln|x-3|+c$$

11. $\dfrac{x+2}{x^3+x}$

$$\frac{x+2}{x^3+x}=\frac{x+2}{x(x^2+1)}$$
$$=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$
$$x+2=A(x^2+1)+(Bx+C)x$$
$$=Ax^2+A+Bx^2+Cx$$
$$=(A+B)x^2+Cx+A$$
$$A=2;C=1;B=-2$$
$$\frac{x+2}{x^3+x}=\frac{2}{x}+\frac{-2x+1}{x^2+1}$$
$$\int\frac{x+2}{x^3+x}dx=\int\left(\frac{2}{x}+\frac{-2x+1}{x^2+1}\right)dx$$
$$=\int\left(\frac{2}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$
$$=2\ln|x|-\ln(x^2+1)+\tan^{-1}x+c$$

12. $\dfrac{1}{x^3+4x}$

$$\frac{1}{x^3+4x}=\frac{1}{x(x^2+4)}$$
$$=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$
$$1=A(x^2+1)+(Bx+C)x$$
$$1=(A+B)x^2+Cx+A$$
$$A=1;B=-1;C=0$$
$$\frac{1}{x^3+4x}=\frac{1}{x}+\frac{-x}{x^2+4}$$
$$\int\frac{1}{x^3+4x}dx$$
$$=\int\left(\frac{1}{x}+\frac{-x}{x^2+4}\right)dx$$
$$=\ln|x|-\frac{1}{2}\ln(x^2+4)+c$$

Q	Learning Outcome***	Example/Exercise	Page
23	**Learn differential equations of the form y'=ky and their general solution**	**(1-8)**	**533**
24	**Solve problems involving differential equations of the form y'=ky satisfying an indicated initial condition**	**(28-30) (31,32)**	**534-535**

In exercises 1–8, find the solution of the given differential equation satisfying the indicated initial condition.

1. $y' = 4y,\ y(0) = 2$

$\frac{dy}{dx} = 4y$

$\frac{dy}{y} = 4dx$

$\ln|y| = 4x + c$

$y = e^{4x+c}$

$y = e^{4x}\,e^{c}$

$y = A\,e^{4x}$ (0, 2)

$2 = A\,e^{4(0)}$

$2 = A$

$y = 2\,e^{4x}$

2. $y' = 3y,\ y(0) = -2$

$\frac{dy}{dx} = 3y$

$\frac{dy}{y} = 3dx$

$\ln|y| = 3x + c$

$y = e^{3x+c}$

$y = e^{3x}\,e^{c}$

$y = A\,e^{3x}$ (0,-2)

$-2 = A\,e^{3(0)}$

$-2 = A$

$y = -2\,e^{3x}$

3. $y' = -3y,\ y(0) = 5$

$\frac{dy}{dx} = -3y$

$\frac{dy}{y} = -3dx$

$\ln|y| = -3x + c$

$y = e^{-3x+c}$
$y = e^{-3x}\,e^{c}$
$y = A\,e^{-3x}$ (0,5)
$5 = A\,e^{3(0)}$
$5 = A$
$y = 5\,e^{3x}$

4. $y' = -2y,\ y(0) = -6$

$\frac{dy}{dx} = -2y$

$\frac{dy}{y} = -2dx$

$\ln|y| = -2x + c$

$y = e^{-2x+c}$
$y = e^{-2x}\,e^{c}$
$y = A\,e^{-2x}$ (0,-6)
$-6 = A\,e^{-2(0)}$
$-6 = A$
$y = -6\,e^{-2x}$

5. $y' = 2y,\ y(1) = 2$

$$\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2dx$$

$$\ln|y| = 2x + c$$

$$y = e^{2x+c}$$
$$y = e^{2x}\, e^{c}$$
$$y = A\, e^{2x} \quad (1,2)$$
$$2 = A\, e^{2(1)}$$
$$2e^{-2} = A$$
$$y = 2e^{-2}\, e^{2x}$$
$$y = 2\, e^{2x-2}$$

6. $y' = -y,\ y(1) = 2$

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -dx$$

$$\ln|y| = -x + c$$

$$y = e^{-x+c}$$
$$y = e^{-x}\, e^{c}$$
$$y = A\, e^{-x} \quad (1,2)$$
$$2 = A\, e^{-(1)}$$
$$2e = A$$
$$y = 2e\, e^{-x}$$
$$y = 2\, e^{1-x}$$

7. $y' = y - 50,\ y(0) = 70$

$$\frac{dy}{dx} = y - 50$$

$$\frac{dy}{y-50} = dx$$

$$\ln|y - 50| = x + c$$

$$y - 50 = e^{x+c}$$
$$y - 50 = e^{x}\, e^{c}$$
$$y - 50 = A\, e^{x} \quad (0,70)$$
$$70 - 50 = A\, e^{(0)}$$
$$20 = A$$
$$y - 50 = 20\, e^{x}$$
$$y = 20\, e^{x} + 50$$

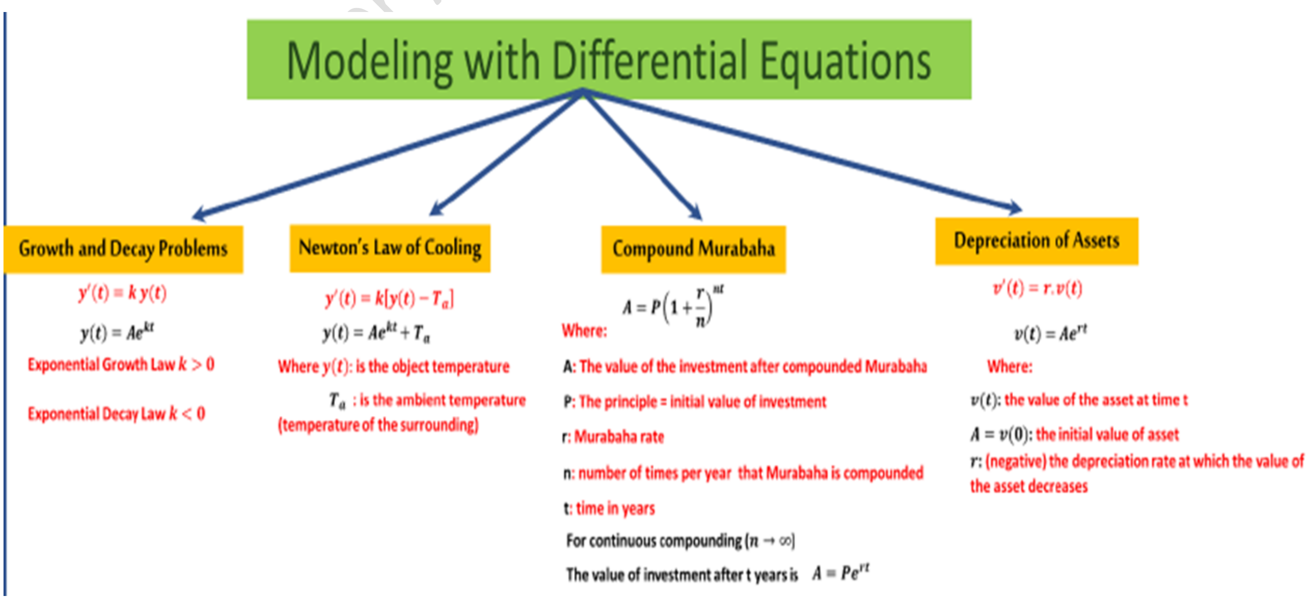
8. $y' = -0.1y - 10,\ y(0) = 80$

$$\frac{dy}{dx} = -0.1y - 10 \quad \ldots\ldots\ldots \quad \frac{dy}{dx} = -0.1(y + 100)$$

$$\frac{dy}{y+100} = -0.1dx$$

$$\ln|y + 100| = -0.1x + c$$

$$y + 100 = e^{-0.1x+c}$$
$$y + 100 = e^{-0.1x}\, e^{c}$$
$$y + 100 = A\, e^{-0.1x} \quad (0,80)$$
$$80 + 100 = A\, e^{-0.1(0)}$$
$$180 = A$$
$$y + 100 = 180\, e^{-0.1x}$$
$$y = 180\, e^{-0.1x} - 100$$



Exercises 29–32 involve compound morabaha.

28. If you invest AED 1000 at an annual morabaha rate of 8%, compare the value of the investment after 1 year under the following forms of compounding: annual, monthly, daily, continuous.

Annual: $A = 1000(1+0.08)^1 \approx \$1080.00$

Monthly: $A = 1000\left(1+\frac{0.08}{12}\right)^{12} \approx \$1083.00$

Daily: $A = 1000\left(1+\frac{0.08}{365}\right)^{365} \approx \$1083.28$

Continuous: $A = 1000e^{(0.8)1} \approx \$1083.29$

29. Repeat exercise 29 for the value of the investment after 5 years.

Annual: $A = 1000(1+0.08)^5 \approx \$1469.33$

Monthly: $A = 1000\left(1+\frac{0.08}{12}\right)^{60} \approx \$1489.85$

Daily: $A = 1000\left(1+\frac{0.08}{365}\right)^{5\cdot 365} \approx \$1491.76$

30. Person A invests AED 10,000 in 1990 and person B invests AED 20,000 in 2000.

(a) If both receive 12% morabaha (compounded continuously), what are the values of the investments in 2010?

(b) Repeat for an morabaha rate of 4%.

(c) Determine the morabaha rate such that person A ends up exactly even with person B. (Hint: You want person A to have AED 20,000 in 2000.)

(a) Person A:
$A = 10{,}000e^{.12\cdot 20} = \$110{,}231.76$
Person B:
$B = 20{,}000e^{.12\cdot 10} = \$66{,}402.34$

(b) At 4% interest:
Person A:
$A = 10{,}000e^{(0.04)20} \approx \$22{,}255.41$
Person B:
$A = 20{,}000e^{(0.04)10} \approx \$29{,}836.49$

(c) To find the rate so that A and B are even,
we solve, $10{,}000e^{10r} = 20{,}000$
Solving gives $r = \ln 2/2 \approx 6.93\%$

31. One of the authors bought a set of basketball trading cards in 1985 for AED 34. In 1995, the "book price" for this set was AED 9800.

(a) Assuming a constant percentage return on this investment, find an equation for the worth of the set at time t years (where $t = 0$ corresponds to 1985).

(b) At this rate of return, what would the set have been worth in 2005?

(c) The author also bought a set of baseball cards in 1985, costing AED 22. In 1995, this set was worth AED 32. At this rate of return, what would the set have been worth in 2005?

(a) Let t be the number of years after 1985. Then, assuming continuous compounding at rate r,

$9800 = 34e^{r\cdot 10}, e^{10r} = \dfrac{9800}{34}$

$r = \dfrac{1}{10}\ln\left(\dfrac{9800}{34}\right) \approx .566378$

Therefore,

$A = 34e^{\frac{1}{10}\ln\left(\frac{9800}{34}\right)t} = 34\left(\dfrac{9800}{34}\right)^{t/10}$

(b) In 2005, $t = 20$ and

$A = 34\left(\dfrac{9800}{34}\right)^2 = \$2{,}824{,}705.88$

(c) The equation for the value of the cards is $y(t) = Pe^{rt}$.

We take $t = 0$ to correspond to the year 1985 which means that $P = 22$.

To determine k we use

$32 = y(10) = 22e^{10r}$

Solving for r gives , $r = \dfrac{1}{10}\ln(32/22)$

The value in 2005 is then given by

$y(20) = 22e^{20r} \approx \$46.55$

32. Suppose that the value of a AED 40,000 asset decreases at a constant percentage rate of 10%. Find its worth after (a) 10 years and (b) 20 years.

Compare these values to a AED 40,000 asset that is depreciated to no value in 20 years using linear depreciation.

With a constant depreciation rate of 10%, the value of the \$40,000 item after ten years would be,

$40{,}000(e^{-(0.1)10}) = 40{,}000e^{-1} \approx \$14{,}715.18$

and after twenty years

$40{,}000(e^{-(0.1)20}) = 40{,}000e^{-2} \approx \$5{,}413.41$

By the straight line method, assuming a value of zero after 20 years, the value would be \$20,000 after ten years.

Q	Learning Outcome***	Example/Exercise	Page
25	**Find the general solution of separable differential equations of first order**	**(5-16)**	**544**

In exercises 5–16, the differential equation is separable. Find the general solution, in an explicit form if possible.

5. $y' = (x^2+1)y$ $\frac{1}{y}y' = x^2+1$ $\int \frac{1}{y}dy = \int (x^2+1)dx$ $\ln\lvert y\rvert = \frac{x^3}{3}+x+c$ $y = e^{x^3/3+x+c} = Ae^{x^3/3+x}$	6. $y' = 2x(y-1)$ $\frac{1}{y-1}y' = 2x$ $\int \frac{1}{y-1}dy = \int 2xdx$ $\ln\lvert y-1\rvert = x^2+c$ $y-1 = e^{x^2+c}$ $y = 1+Ae^{x^2}$

7. $y' = 2x^2y^2$ $\frac{1}{y^2}y' = 2x^2$ $\int \frac{1}{y^2}dy = \int 2x^2dx$ $-\frac{1}{y} = \frac{2x^3}{3}+c$ $y = -\frac{1}{2x^3/3+c}$	8. $y' = 2(y^2+1)$ $\frac{1}{y^2+1}y' = 2$ $\int \frac{1}{y^2+1}dy = \int 2dx$ $\arctan y = 2x+c$ $y = \tan(2x+c)$

9. $y' = \frac{6x^2}{y(1+x^3)}$	10. $y' = \frac{3x}{y+1}$
$yy' = \frac{6x^2}{1+x^3}$ $\int ydy = \int \frac{6x^2}{1+x^3}dx$ $\frac{1}{2}y^2 = 2\ln\lvert 1+x^3\rvert + c$ $y = \pm\sqrt{4\ln\lvert 1+x^3\rvert + c}$	$(y+1)y' = 3x$ $\int (y+1)dy = \int 3xdx$ $\frac{y^2}{2}+y = \frac{3}{2}x^2+c$

11. $y' = \frac{2x}{y}e^{y-x}$

$y' = \frac{2x}{y}e^{y-x}, \quad y' = \frac{2x}{y} \times \frac{e^y}{e^x}$

$y'ye^{-y} = 2xe^{-x}$

$\int y'ye^{-y}dx = \int 2xe^{-x}dx$

$\int ye^{-y}(y'dx) = \int 2xe^{-x}dx$

or $\int ye^{-y}dy = \int 2xe^{-x}dx$

$\int ye^{-y}dy = -ye^{-y} - e^{-y} + c$

and $\int xe^{-x}dx = -xe^{-x} - e^{-x} + c$

$-ye^{-y} - e^{-y} = 2(-xe^{-x} - e^{-x}) + c$

$-ye^{-y} - e^{-y} = -2xe^{-x} - 2e^{-x} + c.$

12. $y' = \frac{\sqrt{1-y^2}}{x\ln x}$

$\frac{y'}{\sqrt{1-y^2}} = \frac{1}{x\ln x}$

$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{1}{x\ln x}dx$

$\arcsin y = \ln(\ln x) + c$

$y = \sin[\ln(\ln x) + c]$

13. $y' = \frac{\cos x}{\sin y}$

$y' = \frac{\cos x}{\sin y}$

$(\sin y)\, y' = \cos x$

$\int (\sin y)\, y'(x)dx = \int (\cos x)dx$

or $\int (\sin y)\, dy = \int (\cos x)dx$

$\cos y = -\sin x + c.$

14. $y' = x\cos^2 y$

$\sec^2 y y' = x$

$\int \sec^2 y dy = \int x dx$

$\tan y = \frac{x^2}{2} + c$

$y = \tan^{-1}\left(\frac{x^2}{2} + c\right)$

15. $y' = \frac{xy}{1+x^2}$

$\frac{1}{y}y' = \frac{x}{1+x^2}$

$\int \frac{1}{y}dy = \int \frac{x}{1+x^2}dx$

$\ln|y| = \frac{1}{2}\ln|1+x^2| + c$

$y = e^{\frac{1}{2}\ln|1+x^2|+c} = k\sqrt{1+x^2}$

16. $y' = \frac{2}{xy+y}$

$yy' = \frac{2}{x+1}$

$\int y dy = \int \frac{2}{x+1}dx$

$\frac{y^2}{2} = 2\ln|x+1| + c$